Navigating through the mathematical world: Uncovering a geometer's thought processes through his handouts and teaching journals

Sepideh Stewart [1], Clarissa Thompson[2] and Noel Brady[3]

[1]University of Oklahoma, Department of Mathematics, Oklahoma, USA; sstewart@math.ou.edu

[2]Kent State University, Department of Psychological Sciences, USA; cthomp77@kent.edu

[3]University of Oklahoma, Department of Mathematics, Oklahoma, USA; nbrady@ou.edu

In this case study, we examined a mathematician's thought processes as he taught a course on Algebraic Topology. The mathematician shared his teaching-related journals over an entire semester and discussed them in depth during weekly meetings with the research team comprised of a mathematics educator, a cognitive psychologist, and a postdoctoral fellow in mathematics. Concurrently, one of his students took detailed journals on most lectures. The team employed Tall's three worlds of embodied, symbolic, and formal mathematical thinking as various lenses to gain insight into the mind of the working mathematician as he taught a course on Algebraic Topology. Although, the analysis of the data from the instructor's journals and the in depth discussion of the journals during the team meetings revealed his thought processes, the 35 handouts that he prepared, aligned with students' needs, provided the most insight into his way of thinking.

Keywords: Embodied, symbolic, formal, Tall's three worlds, Algebraic Topology

Introduction

Communicating advanced mathematical ideas to university students is a challenging endeavor. It is a common and accepted practice for many mathematicians to write definitions, theorems and proofs on the board and make comments as they introduce mathematical ideas to students. Thurston (1994, p. 162) asked the question: "How do mathematicians advance human understanding of mathematics?" In interviewing 70 research mathematicians, Burton (1999, p. 31) found that "intuition, insight, or instinct" was seen by most mathematicians as a necessary component for developing student knowledge. Although, we have some literature on examining mathematicians teaching practices (e.g. Fukawa-Connelly, 2012; Stewart, Schmidt, Cook & Pitale, 2015), research on what takes place in the minds of mathematicians and their students is still scarce (Speer, Smith, & Horvath, 2010). Dreyfus (1991) believed that, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician" (p. 29). In this study, we examined a mathematician and one of his students' daily thoughts on Algebraic Topology. The overarching goal of this research was to investigate the way mathematicians and students think about mathematics and the possible pedagogical challenges that they may face.

Theoretical Framework

In this study, we employed Tall's (2013) three-world model of conceptual embodiment, operational symbolism, and axiomatic formalism in order to describe an expert geometer's ways of mathematical thinking. In Tall's view, the embodied world involves mental images, perceptions, and thought experiments; the symbolic world involves calculation and algebraic manipulations; the formal world involves mathematical definitions, theories and proofs. Tall (2008) asserts that, "all humans go through a long-term development that builds through embodiment and symbolism to formalism" (p. 23). Bridging between the embodied and symbolic worlds is of critical importance. Tall emphasizes that "a curriculum that focuses on symbolism and not on related embodiments may limit the vision of the learner who may learn to perform a procedure, even conceive of it as an overall process, but fail to be able to imagine or 'encapsulate' the process as an 'object' (p. 12).

Tall and Mejia-Ramos (2006, p. 3) declared that the word 'world' is carefully chosen and has a 'special meaning' in order to represent "not a single register or group of registers, but the development of distinct ways of thinking that grow more sophisticated as individuals develop new conceptions and compress them into more subtle thinkable concepts". As Dreyfus (1991, p. 32) declares "One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take… Teaching and learning this process of switching is not easy because the structure is a very complex one." Duval (2006) claims that many students do not have the cognitive framework to perform the switch.

Tall's theoretical framework accounts for movement between the three worlds (e.g. embodying the symbolism and symbolizing the embodiment). However, careful research into the ways that mathematicians move between modes of thought and facilitate their students' movements among these modes of thought are still scarce in the literature. We endeavored to investigate the following questions in this study: (a) How did the instructor and student move between the formal, symbolic, and embodied worlds? (b) How did the instructor use handouts in order to help students move between the worlds?

Viewing Homology Theory through three lenses

The mathematician appreciated the developmental aspect of Tall's framework in which one begins with a very embodied view of the world around them and then moves with increasing age and experience to a symbolic view as one matures. However, he took issue with the "formal" viewpoint as the ultimate destination of this progression, especially since formal from a math perspective (i.e., set theoretic axioms, definitions, and formal deductions from such a system) is not the way mathematicians think. One can program a computer to generate (i) statements and (ii) formal proofs of these statements within an axiomatic system. In what sense can we say that the computer is discovering a mathematical theory? Humans use a lot more when they discover/develop a mathematical theory. Among all the myriad of possible statements that could be true in this formal theory, mathematicians choose certain ones (usually as a result of intuition and metaphors possibly supported by symbolic computations to garner evidence for the particular statements) called conjectures, and they try to prove them. Instead, the mathematician made sense of Tall's worlds by thinking of them as three lenses capable of viewing a mathematical reality/world. Figure 1 illustrates his views of Homology Theory through these lenses. The embodied lens sees cycles as geometric objects, and similarly for chains and various topological spaces. The symbolic lens uses symbolic computation tools such as the Mayer-Vietoris sequence and produces symbolic computations (e.g., the homology of the 2-torus). The formal lens works with the Eilenberg-

Steenrod axioms and results which can be derived formally from these axioms. The geometric side of topology spans the embodied and symbolic lenses. Algebra, primarily in the form of Homological Algebra, spans the symbolic and formal lenses.

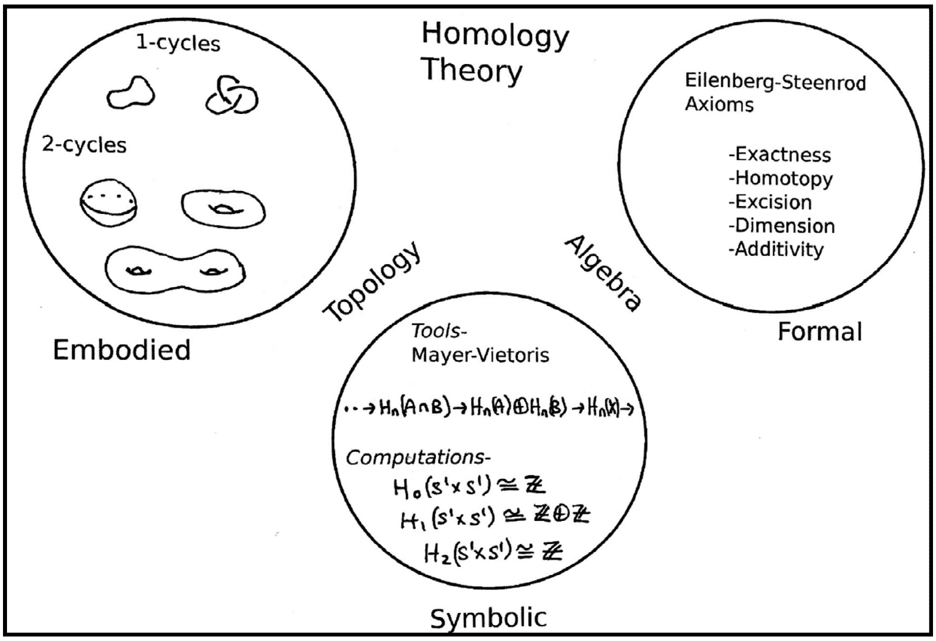

Figure 1: The three lens view of Homology Theory

We can think of similar lenses, for example, in medicine. One can look at a patient with one's eyes, take an x-ray or an MRI of the patient, view the patient through an infrared lens, listen to the patient's heart and lungs etc., talk to the patient about their symptoms, and draw blood and perform tests. These are all different ways of obtaining information about (getting a more complete picture of) the patient.

Method

The Participants. Our qualitative case study investigated the ways an expert mathematician navigated among Tall's worlds of mathematical thinking. The research team consisted of four members: a mathematics education researcher, a geometer, Noel Brady (the course instructor), a cognitive psychologist, and a mathematics postdoc familiar with Algebra as well as Topology.

The Course. The Algebraic Topology course was the first in a two-semester sequence of graduate courses. There were eight graduate students enrolled in the course. During class meetings, Noel often passed out handouts to help students follow along with the topic of the day. He believed some topics covered in the chosen textbook (Hatcher, 2001), needed to be handled in a more detailed fashion. "Hatcher is a bit fast and loose with all of this". Students actively solved problems together in groups, or individual students were called to the board to complete problems. Noel also helped to revive an extracurricular, student-led topology seminar.

Data and Procedures. In this study, we analyzed a geometer's thought processes and actions while he taught Algebraic Topology over the entire Fall 2014 semester. One source of data was a series of teaching journals that contained Noel's reflections on his preparations for class, what happened during class, as well as some descriptions of the events that took place during office hours and a student-led topology seminar. The research team read his daily journal entries and discussed them during weekly research meetings. During these meetings, we asked Noel further clarification questions, and he often drew additional pictures as he described the course content. These meetings were audio recorded and later transcribed and will be used as our source of data. Another source of data came from one of Noel's graduate students who also wrote daily journals. These student journals provided an additional perspective into the events that took place in class. In addition, the data came from the handouts (35 in total) that Noel provided. So far, we have only coded Noel's teaching journals.

Coding Scheme. In addition to assigning codes for the three worlds of embodied, symbolic, and formal mathematical thinking, we also created codes (see Table 1) for movement between the worlds (e.g., embodied-symbolic).

Results and Discussion

Table 1 shows the percentage of total qualitative codes that were applied to excerpts from Noel's teaching journals. The main theme of Tall's three worlds of mathematics comprised 25% of the total codes. Teaching was the main theme that was coded the most (46%) in Noel's journals. Reflections included 20% of codes, and codes pertaining to students involved 9% of the total codes.

Main Theme	Sub-Category	Percentage Use Main Theme	Percentage Use Sub-Category
Teaching		46%	
	Pedagogy		26%
	Examples		22%
	Handouts/notes		20%
	Proofs		11%
	In-Class Activities		10%
	Hatcher/textbook		7%
	Theorems		3%
	Homework		2%
Tall's Worlds and Movements		25%	
	Symbolic		25%
	Embodied (Intuition)		21%
	Topology - Algebra		20%
	Embodied-Formal		18%
	Formal		10%
	Embodied-Symbolic		7%
	Symbolic-Formal		0%
Reflections		20%	
	On students: understanding/preparedness		47%
	On self: Careful/precise		31%
	Pacing		22%
Students		9%	
	Igor (one of the students)		26%
	Students ask questions outside of class		26%
	Student seminar		26%
	Students ask questions in class		22%

Note: Percentage Use Main Theme adds to 100%
Percentage Use Sub-Category adds to 100% for each sub-category

Table 1: Qualitative coding scheme

Analysis of the data revealed ample evidence that Noel repeatedly navigated between the three worlds of mathematical thinking. Below, we provide examples from our analysis of his teaching journals and a student's journals to illustrate movement between the embodied, symbolic, and formal worlds.

Moving between Embodied (Intuition) and Formal Worlds

According to Noel, this may have been the type of movement that the students found the most challenging. *There were a lot of questions about how to pass from an intuition to a formal proof (many of these examples used techniques/results from quotient spaces).*

The analysis of the student's journals showed his concerns regarding the proofs. This excerpt was taken from one of his journals at the beginning of the semester:

Dr. Brady's way of proving results that come from concepts we're already supposed to have come across before his class is nice, I think. He gives a detailed outline verbally, which is helped along visually by his pictures and hand gestures. For the most part I'll watch without writing almost anything, but I definitely get a lot out of reviewing concepts in this way. I'm a little worried, however, that when we get to brand new material Dr. Brady's way of proving results might remain in the same verbal/hand-waving/picture-drawing style and that this won't be enough for me to follow the proof right there and then. He tends to speak and write very quickly, which is fine when we're reviewing. But since I can either copy furiously what he writes on the board or listen to him, but not both, this could become a problem.

Noel refused to give students proofs that were pre-packaged. More specifically, he wanted to provide students with intuitions/pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it. In one of the research meetings Noel said:

I mean I can give verbatim proofs of things or give them more detailed proofs where Hatcher leaves stuff out, but that will just waste time and I'll reproduce a book and nobody will get anything out of it. So I've given them intuitions, enough of an intuition that they can tag that together with a formal proof.

Later in the course the student wrote: *I've seen van Kampen's theorem before, but Dr. Brady's from-the-ground-up approach was very nice in that it showed us through comprehensive diagrams just where exactly the theorem comes from.*

Movement between Embodied and Symbolic Worlds

Noel discussed moving from embodied demonstrations (e.g., rope trick) to having students complete symbolic examples (e.g., right-angled Artin group (RAAG) complexes and the torus knot spine):

More of the same. I connected back to several examples from the first week and from the intro to π_1. The pair of circle links in S^3 example (a.k.a. the rope trick) and the RAAGs. This seemed to go ok. Mentioned again that RAAGs are deceptively simple looking groups, but that their subgroup structure is surprisingly rich. In particular, Bestvina-Brady (1997) and Agol-Wise (2012) contain very surprising results about subgroups of RAAGs. Told them that the story is still ongoing. Left off with an example of a torus knot spine (Hatcher).

The Handouts

Analysis of the 35 handouts that Noel created illuminated the motives behind some of his thought processes and movement between worlds. These handouts gave the team a more authentic glimpse into the mind of the mathematician than the teaching journals that Noel regarded as self-critical (self-aware). Figure 2 shows the first two pages of a handout Noel created on barycentric subdivision. The start of the handout contains the formal definitions of "barycenter" and of "barycentric subdivision." These definitions build on a previous definition (and square bracket notation) of an n-simplex. The definition of "barycentric subdivision" is recursive (i.e., defined in terms of lower dimensional versions of itself). The rest of the two pages is devoted to building students' intuitions for these definitions. At the bottom of the first page, two embodied examples are provided which demonstrate how to unwrap the recursive definition to determine the barycentric subdivision of a 1-simplex (a line segment) and of a 2-simplex (a triangle). This is followed by an exercise which asks the student to add another layer of recursion and describe the barycentric subdivision of a 3-simplex (a triangular-based pyramid). This is a very embodied example. At this stage, Noel hoped that the student should be gaining confidence working with the recursive definition and should be developing an intuition that the symbolism will work in higher dimensions where one's embodied intuition fails. The second exercise asks the student to iterate the barycentric subdivision process for a 2-simplex. Again this is very embodied and can be drawn easily in the plane. Noel pointed out that developing an intuition about iterated barycentric subdivisions is important since they will form the heart of the proof of the "locality result" and the proof of the "excision theorem" for singular homology later on in the course. The two Roman numeral labeled observations at the end of page 2 build on the student's embodied intuition of the behavior of iterated barycentric subdivisions in dimension 2 (obtained from doing exercise 2). They motivate the statement of the theorem that will be given and proven on subsequent pages of the handout. They also alert the student to the fact that some care will have to be given to the proofs on the subsequent pages. This is particularly so, since these proofs will hold in arbitrary dimensions.

Noel pointed out that, from a textbook perspective, one can skip straight from the definitions of barycenter and barycentric subdivision to the statement and proofs of the theorems about the behavior of the diameters of simplices under iterated barycentric subdivisions. Nothing in the logical progression and framework would be lost. However, students' intuitions would be lacking (save for the rare student or two who can do some mental exercise equivalent of the examples, exercises and observations of these two pages.). This handout is one of a sequence of three handouts. These handouts get increasingly symbolic and abstract. Eventually, the results contained in the last handout are just what are needed in the formal proof of the "locality theorem" (and the "excision theorem") of singular homology. At this stage, the proofs are very symbolic and far removed from geometry. So it is good that students have developed an embodied intuition about iterated barycentric subdivisions, so that they have concrete models in their mind for how excision works on the geometric level of chains.

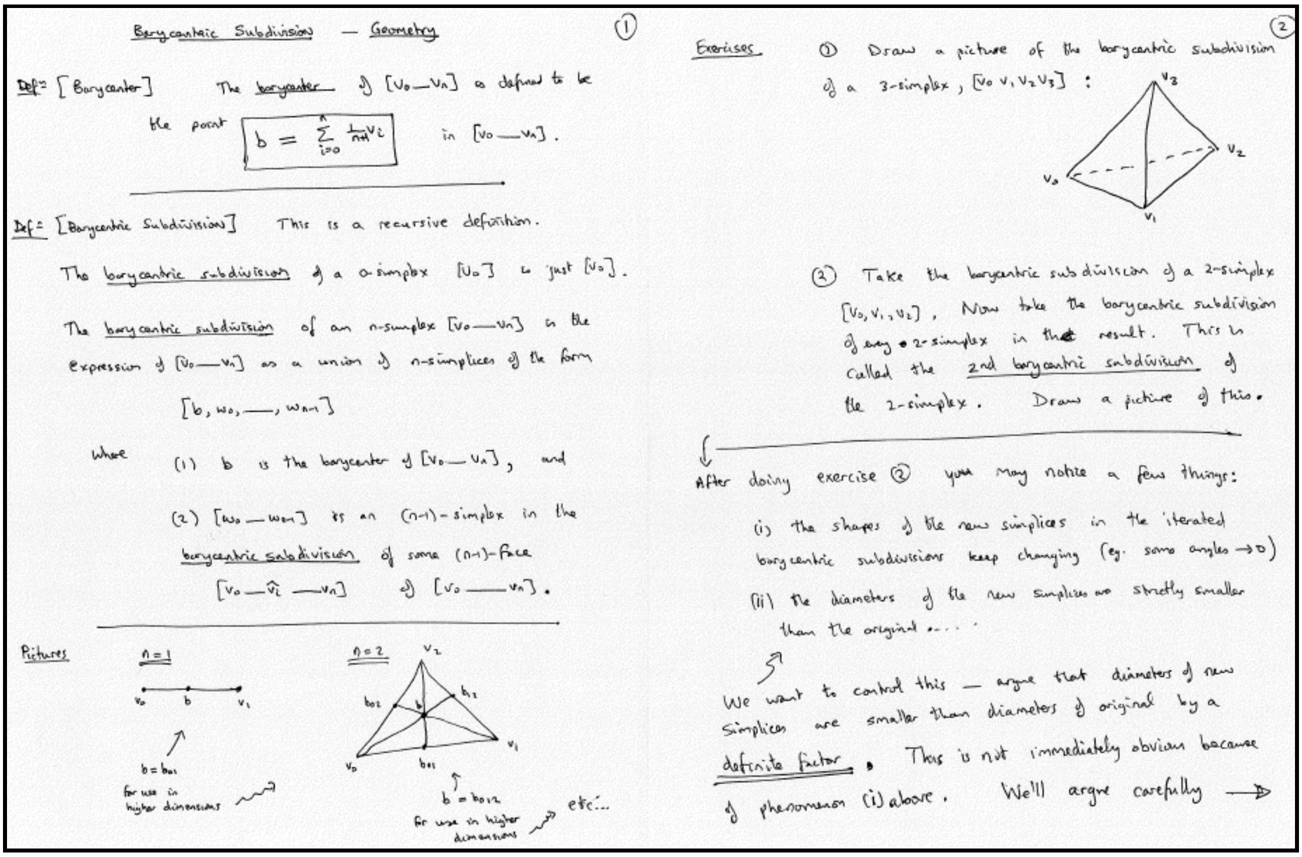

Figure 2: An excerpt from Noel's handout

Concluding Remarks

This study revealed that Noel viewed Algebraic Topology through all three mathematical lenses (embodied, symbolic, formal), and his handouts provided his students with opportunities to view the course material through these different lenses as well. In one of the research meetings, Noel mentioned: *When I think of the mathematical world of algebra I have examples in my mind, many of which are very embodied, and many of which are symbolic, I also know the axiomatic definitions of concepts in this world like "group," "ring," "field" etc. So when I think of the world of algebra all three lenses (EMBODIED, SYMBOLIC, FORMAL) kick into gear. Likewise, for the mathematical world of topology.*

Our research team, comprised of a mathematician, a mathematics educator, and a cognitive psychologist, are working together to apply and evolve Tall's theoretical framework by analyzing the teaching journals of mathematicians and their students. We have come to realize that the embodied, symbolic, and formal worlds blend together as applied to Algebraic Topology; it is often not clear where one world starts and another world ends. In addition to thinking about problems from the ESF perspectives, mathematicians often translate a problem from one area of mathematics (e.g. Topology) to another (e.g. Algebra). This translation is achieved through the use of mathematical constructs called *Functors*.

Noel used the analogy of a translator to describe the mathematical notion of a functor. When a statement of a problem is translated from one language to another, some of the details may get lost in the translation. Perhaps this loss of information has an unexpected benefit; the simpler formulation of the problem in the new language might allow for new insights or intuitions to be gained, and perhaps even for a solution to the original problem.

Noel talked about functors in his journals, and described how they are used to solve problems in topology by first translating then into algebra problems.

We introduced some other situations where Algebraic Topology functors might help solve topology problems, and mentioned that the homology functors would be introduced and studied in the course.

We are using analogies and metaphors to communicate with one another as we attempt to understand the pedagogical decisions of the working mathematician. As Thurston (1994, p. 168) asserted: "we mathematicians need to put far greater effort into communicating mathematical ideas. To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking...we need to appreciate the value of different ways of thinking about the same mathematical structure".

References

Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics*, *19* (3), pp. 27-32.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 25-41). Dordrecht: Kluwer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103-131.

Fukawa-Connelly, T. (2012). A case study of one instructor's lecture-based teaching of proof in abstract algebra. *Educational Studies in Mathematics*, *81*(3), 325-345.

Hatcher, A. (2001). *Algebraic Topology*. From:www.math.cornell.edu/~hatcher/AT/AT.pdf

Speer, N. M., Smith, J. P & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior*, *29*, 99–114.

Tall, D. O., & Mejia-Ramos, J. P. (2006). The long-term cognitive development of different types of reasoning and proof, presented at the Conference on Explanation and Proof in Mathematics: Philosophical and Educational Perspectives, Essen, Germany.

Tall, D. O. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*, 5-24.

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Thurston, W. (1994). On proof and progress in mathematics. Bulletin*(New Series) of the American Mathematical Society*, 30 (2), 161-17