

Historical Methods for Drawing Anaglyphs in Geometry Teaching

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The historical anaglyphic method was in use for more than hundred years to create spatial illusions of mathematical objects and for technical constructions. While algebraisation is predominant at school, students lack experience in understanding the causalities of technical tools. Modern technical devices rarely allow for direct investigations on underlying technical principles. Here we use the historic anaglyphic method to enable the students to produce high-standard 3D illusions just by using coloured pencils, a ruler and glasses with colour filters. We developed an approach to the anaglyphic method that uses nothing but similarity and especially surpasses projective geometry. The presented approach relates plane and spatial geometry, and can be grasped by all students that have some understanding of the similarity of triangles.

Keywords: Anaglyph, binocular geometry, historical methods of visualization, 3D-representations

Introduction

Concept development in high school mathematics – in particular A-level subjects in many countries – is characterized by predominant algebraisation. This seems to match successful and effective heuristics, attitudes and problem-solving strategies in modern everyday lived experience. Out of school it fosters the learning of pattern recognition, algorithmic procedures and trial and error. The progressive digitalisation of most fields of experience and actions of contemporary students seems to make the search for causalities and functional principals superfluous and unnecessary. Modern technical devices rarely for allow direct investigations on underlying technical principles. However, approaches and attitudes aiming only for the use and not the understanding of technical devices involve the risk of simple manipulation of the user and restrict creative developments of the tools by their users. In order to help our students to become autonomous, mature individuals, we need in mathematics (and also as a foundation for technical sciences) teaching designs where students start to wonder: How does it work? How can I accomplish it on my own?

In the following we present materials for discovery learning, supporting the described educational goals. The design of teaching and learning materials allows for the implementation of the history of engineering and technical drawing in different ways and with different aims. A historical investigation using original sources can be conducted as an introductory part of a workshop on 3D presentations. Historical investigations can also be undertaken as a part of an individual student's presentations after the workshop on anaglyphs and binocular geometry. We start with a short historical introduction into anaglyphs as well as into the literature and original sources, which are applicable by the students at school or by student teachers in teacher education.

We then give a short summery of how binocular geometry can be related to the mathematics curriculum and discuss a textbook presentation of the topic. The following section deals in detail

with the geometrical properties of central projections, which enable the drawing of anaglyphs without using three-dimensional analytic geometry.

We tried out the developed set-up and the material in four workshops at an international Kangaroo Camp, each with 15-20 high school students between the ages of 15-18, and in a day course at the Hausdorff Center with local high school students in Bonn, with a lecture and problem sessions with about 60 high school students.

Historical Background

Sir Charles Wheatstone invented the earliest type of stereoscope in 1838 (Wheatstone, 1838). However, as David Brewster writes in (Brewster, 1856, p.27) a certain Mr Elliot was led to the study of binocular vision ... as early as 1823. Wheatstone and Elliot used mirrors, whereas Brewster invented lenses. The mathematician Wilhelm Rollmann (1853) invented the anaglyph stereographic method. The Greek word *Anaglyph* is derived from *áná*(aná), meaning “on” or “on each other“ and *γλύφω*(glýphō) “to carve”, “engrave”, or “represent”. In his method, two pictures in the mutually different colours blue and yellow are superposed onto each other. The observer in Rollmann’s method separates these pictures by using glasses with colour filters, i.e. a red glass for the left eye and a blue one for the right eye.¹ All cited sources are digitalised and available online. The descriptions of the methods are mainly verbal. The few calculations can be performed and understood using elementary middle school mathematics. This makes the cited literature a good and suitable source for historical investigations in the classroom.

Rather than being a goal as such, the anaglyph method was already used by Rollmann to illustrate mathematical facts and insights in a three-dimensional fashion. Another impressive example of this kind of illustration is the Imre Pal’s beautiful book (1961). Although the books of Rollmann and Pale describe the three-dimensional model of spatial seeing, nevertheless they do not instruct how to draw two-dimensional anaglyph stereographic pictures. For the latter we designed a special workshop, which can be related to schoolbook exercises in analytic geometry (Körner et al., 2010, p. 92) or can serve independently for project work.

Anaglyphs in Teaching

Binocular geometry is not a canonical topic for mathematics lessons. So we were rather surprised to find a set of exercises related to the anaglyph method as well as materials for teacher training courses. The following excerpts in Figure 1 are taken from a German school textbook for grade 11 (one year before A-levels). The book is one of the five most widely used textbooks in high schools in Germany. The excerpts stem from a chapter in analytic geometry dealing with the calculation of intersections of lines and planes given in Cartesian coordinates. In addition to perspective drawings of a house and bowling pins, some general properties of perspective drawings are given (left). In preparation for the work with perspective mappings, the textbook authors posed the exercise to compute the image of the shadow of a cube illuminated by a lamp given in Cartesian

1 In our case, we use a red glass for the left eye and a blue or better turquoise one for the right eye.

coordinates. The shadow has to be calculated in a given plane using the formulas for intersection points of lines with planes. The green image (right) is the cube shadow in the given plane. The red image is the result for the same calculation but with a slightly shifted illuminant. The students are asked to make red-green glasses themselves and to look at the picture through these glasses. Among the textbook materials for computer-based learning, there are 3D dynamic geometry applets with the green and the red calculated images corresponding to different dynamic light sources. Even though the exercises deal with a geometrical context – perspective drawings – the proposed approach is purely algebraic.

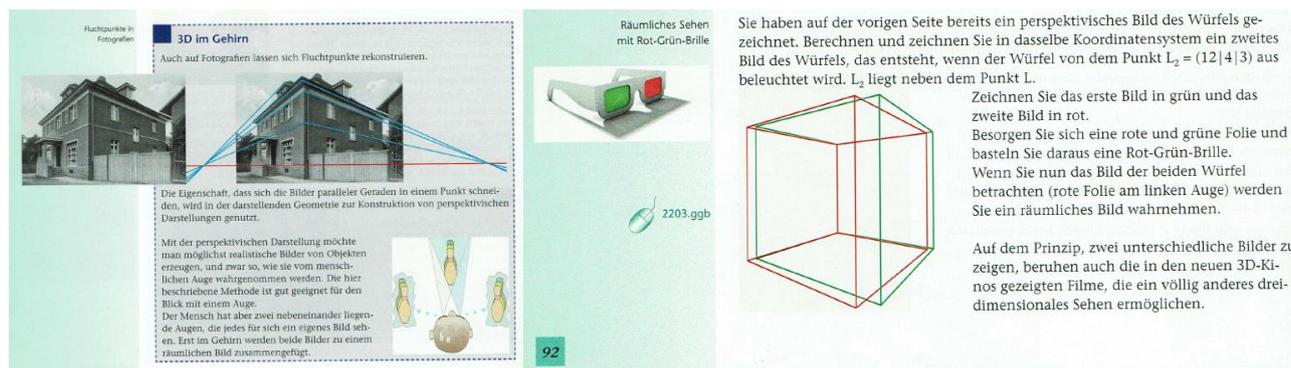


Figure 1: Excerpt of the textbook series *Neue Wege*.

The pictures are visualizations of the results of algebraic calculations. In (Färber, 2016) the author proposes designing the excursion on the anaglyph stereographic method as discovery learning in groups. Nevertheless, the lesson planning only involves algebraic manipulations. As we already discussed in (Kaenders & Weiss, 2016), the high degree of abstraction and technical complexity of algebraic symbolic language gives students few opportunities to question the underlying rules, to introduce their own situated notations and notions reflecting their individual understanding of a problem and its context or to develop their own mathematical questions. To deal with this problem we develop a geometrical context for discovery learning of the anaglyph stereographic method.

A simple approach: lifting plane figures

When we considered using the historical anaglyph method for teaching, we expected a couple of difficulties. First, we were convinced that we would have to give a quick introduction to projective geometry and then apply it to the anaglyph method. We were then surprised to discover an approach to the fabrication of such binocular illusions that does not use projective geometry at all. It is the technique of *lifting a point*. Well understood, it allows not only for the lifting of points but also the lifting of any figure from the plane as long as it is supposed to become an illusion of a congruent figure in a plane parallel to the initial plane. We give a description of the course of action in the workshops.

Practical preparation

Before the students are given the task of creating their own anaglyphs, we show them some exemplars. By looking through the red-green-glasses they begin to get a feeling for what such a painting could look like and which type of effects are generally possible. It turned out to be a helpful practical hint to let the student put their fingertip on the spatial spot where they expect the

figure to be. A few students do not succeed in recognizing the intended illusion. The reason for this might be problems like shifted eyesight, where one eye has a stronger visual faculty than the other, or a red green deficiency. Nevertheless, these students could successfully participate in the workshop.

Erecting a little stick

The first exercise is to erect virtually a little stick of thumb size to an illusion that appears to be an orthogonal stick on the paper. In a second step, we can also let the stick appear slightly levitated. In order to find such a representation of the desired illusion, the students can turn the question around: the stick is given and we seek the red-green drawing on our paper. If we put a stick (like a pencil) orthogonally on the paper, we can conceive the red and the green drawing as the shadow we obtain when we imagine our eyes to be light sources.

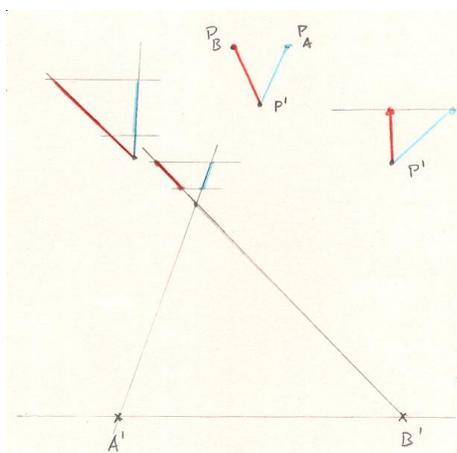


Figure 2: Erecting a stick of thumb size by lifting a point.

Almost all students draw a short red and a short green line segment that produce two lines that intersect at the point that is ought to be the orthogonal projection of the stick on the paper. By discovering this, two questions arise:

- What is the angle between the red and the green line segment? The students relate it to the position of the eyes and some conjecture that the lines prolong to the pedals of the eye points.
- How far do we have to draw the line segments for a perfect illusion? After having explored the situation, the students give two conjectures: First, the ratio of the red and green line segment is the same as the ratio of line segments between the pedal points of the eyes and the point where the stick touches the paper. Second, the line $A'B'$ is parallel to P_AP_B . They indicate that because of similarity, both assertions are equivalent.

Analysis of the exercise

To analyse the situation, we assume the position of the eyes A and B on fixed height h over the table. We assume the eyes A and B to be parallel to the table plane and to have a distance of 7 cm, which is about the average eye distance for adult human beings. When we project the eyes A and B orthogonally onto the table, we obtain two pedal points A' and B' . We consider an arbitrary point P'

in the plane. Now we want to find points P_A and P_B in the plane with the effect that they create the illusion of a point P in space, that *lifts* our point P' to some height h .

In Figure 3 we see that this illusion arises when the lines AP_A and BP_B cross in the point P . Then ABP forms a plane. Then the three planes $A'B'BA$, $A'B'P$ and ABP have three intersection lines, two of which are parallel. Then the third one is parallel as well. The reason is what one can call the *Theorem of the Tent*: Given three planes E_1, E_2 and E_3 that intersect in three lines $g_{12} = E_1 \cap E_2$ and $g_{23} = E_2 \cap E_3$ as well as $g_{13} = E_1 \cap E_3$. When two of these lines are parallel to each other, the third is parallel to both as well. The proof of this proposition is an occasion to show the efficiency of the set theoretic language: Assume that the lines $g_{12} = E_1 \cap E_2$ and $g_{23} = E_2 \cap E_3$ are not parallel; they then intersect in a point P , since both lie in a plane. Then $\{P\} = g_{12} \cap g_{23} = E_1 \cap E_2 \cap E_3$ and $g_{13} = E_1 \cap E_3$ intersect in P as well.

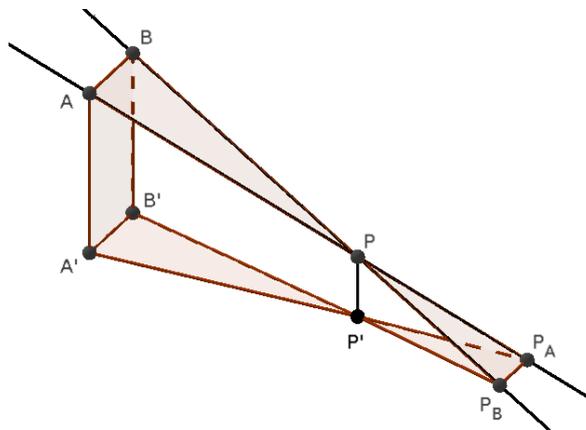


Figure 3: The basic principle of point lifting.

Hence, we can *lift* the point P' to an illusionist point P , when we draw the lines $A'P'$ and $B'P'$ and end up at points P_A and P_B , such that P_AP_B is parallel to $A'B'$. In order to find out how far the point will be lifted, we use the point of view of similarity as the students uttered it. We consider one of the two triangles $A'P_PA$ or likewise $B'P_PB$ (see Figure 4). We especially want to understand the relation between the height h and the distance $d := P_AP_B$.

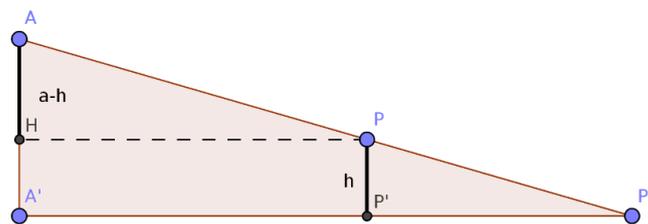


Figure 4: One of the triangles of the basic figure.

In Figure 4 we read off the following ratios: $\frac{a}{h} = \frac{A'P_A}{A'P'}$ and $\frac{a-h}{h} = \frac{AP}{PP_A}$. Combining this with the ratio between the triangles ABP and PP_AP_B , we conclude $\frac{7}{d} = \frac{AP}{PP_A} = \frac{a-h}{h}$. Thus $d = \frac{7h}{a-h}$ or $h = \frac{ad}{7+h}$.

Note the remarkable fact that the distance d does not depend on the special position of P' . Hence, we have one method to lift not just one point but also a whole figure to a certain fixed height h . For

instance, we can lift a square by lifting its vertices on the same height and then connect the corresponding red and green points. If we do that twice, we can construct the illusion of spatial box.

Similarity as key concept

Finally, we want to understand how we can lift a figure that does not consist of line segments, e.g. a circle. For this, we need to understand how to lift an arbitrary figure to a fixed height h . For this fixed h , we consider the map of the plane to itself, that maps P' to P_A and likewise the map that maps P' to P_B . We know already the answer, since $\frac{A'P'}{A'P_A} = \frac{a}{h}$. Therefore, both are central dilations with the factor $\frac{a}{h}$, one with centre A' , and the other with centre B' .

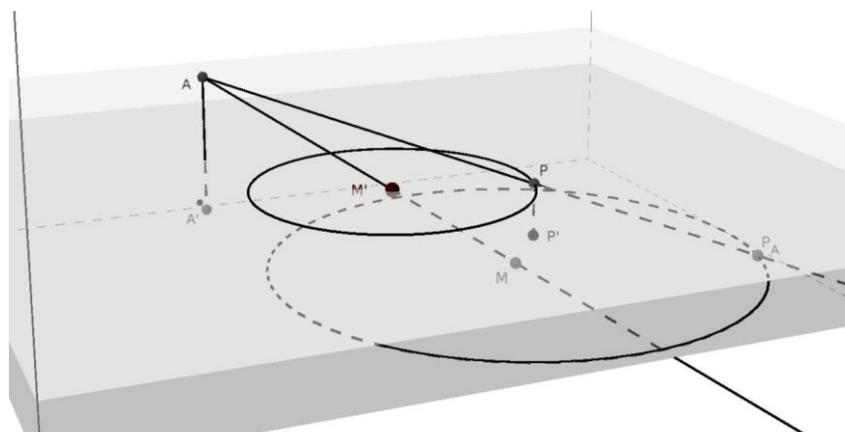


Figure 5: Central dilation with factor $\frac{a}{h}$.

These considerations on similarity can be used to construct tasks for instructional scaffolding as well as materials for explorative learning.

Observations during the workshops and development of research questions

In our workshops, we developed most of the tasks and drawings together with the students on the blackboard. This gave us the possibility of choosing between small-step guiding tasks and rather open activity-oriented tasks corresponding to the work of the students. The groups in the four workshops at the international Kangaroo Camp were very inhomogeneous regarding their English language skills as well as their mathematical preparation. These groups had students interested in mathematics but without any training for competitions or mathematical extracurricular experiences and participants of the International Mathematical Olympics. None of the participants had ever dealt with binocular geometry. In spite of their age (15-18 years old) there was no problem to get the students to draw pictures with crayons. It became also evident that the problem 'How to draw an anagram' is extremely suitable for inhomogeneous groups. The students worked in small groups and were quickly fascinated by their own experiments and pictures. Our objective that was reached in all four workshops was to get the students to search for the underlying principles of constructions and to produce their own drawings by using the principles. As soon as the students are acquainted with the technique of point lifting, there are many possible projects to tackle.

The day course in Bonn was organized differently. The workshops were held by mathematics teacher students. We gave introductions into anaglyphs first for the tutors than for the students,

attended the different workshops and moderated the presentations of the results. For the day course in Bonn, we prepared a script for the tutors with problems they were supposed to solve. Before the day course there were several meetings where the tutors could ask questions and discuss the concept. There was a noticeable difference between tutors who tried to grasp on their own the concept of drawing anaglyphs by lifting points and curves using the script more to look up some of the details and tutors who first read the script and tried to solve the problems by using the methods described there. The first type of tutors led their workshops in a more explorative experimental way; some of the second type tutors had inserted into their workshops small lectures on the basis of the script. The students asked questions from different perspectives: from a phenomenological perspective (What are conditions for two points to create the illusion of a floating point?), from the perspective of geometrical invariants (Which properties do not depend on the position of the centre of the projection?), from the perspective of geometric transformations (How to place the figure to support its three-dimensional illusion?).

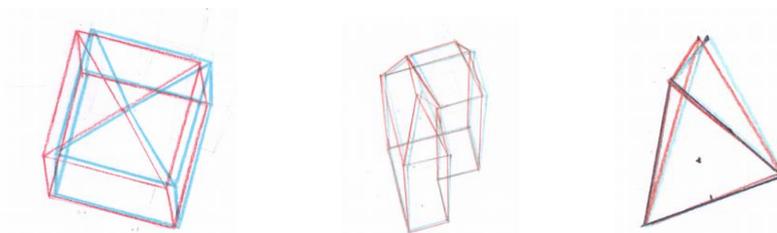


Figure 6: Examples of student products: a box with concave top, a house, a tetrahedron.

Nevertheless, the students of all groups of the day course were very motivated to understand how to draw a 3D picture, experimented with pencils, lifted points and produced their own 3D images (Fig.6). During our reconsiderations of the first repeated workshops, we tried to describe the atmosphere when the students started to construct central projections and calculate distances. At the beginning - may be because of our technical explorations into the past - we called it the mind of engineers and inventors. During the next workshops non the less we realized that a substantial part of workshops the students were engaged with thought experiments: projections through transparent tables, cutting up spheres, building houses out of cubes and tetrahedrons, rotating trefoil knots. This led us to think about the role of thought experiments in physic and mathematic lessons. May be it was so easy to inspire so different students for experiments because there were not real experiments? Students meet nowadays in their everyday life not often people who both are said to be cool and decompose technical devises in order to understand basic functional principles. Choosing advanced courses in science for A-levels does not imply one has grown up with a soldering iron or a chemistry box.

One could think that our students good performance and experiences in virtual worlds could give the historical extremely important thought experiment a new place in physics lessons in order to develop interest in functional principles even if there are only very limited prior technical experiences. But at least in German physic lessons, it seems to be rather the other way around: Many computer based visualisations and experiments transform the very nature of the thought experiment and replace it by a confirmation experiment of a virtual programmed reality (as it is also done in the earlier discussed mathematics textbook task using the applet to compute the anaglyphs).

May be it became rather a task for mathematics educators to integrate thought experiments and virtual engineering into mathematics teaching to support the perspective: How does it work?

Conclusion

The fact that the participants in our workshops were able to pose independent research questions in geometrical terms with geometrical meaning is for us an indication of their development of a geometrical language and concept. The students gave their proofs by construction and in different geometrical notions using invariants and similarity mappings. We were especially impressed by the ability to switch between plane and spatial perspectives on the stereographic pictures developed by the participants during the workshops.

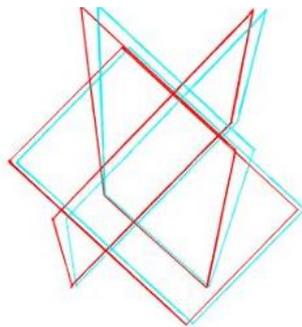


Figure 7: Borromean rings made of three golden rectangles.

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Acknowledgment

The authors thank Alexander Unger from the Humboldt University Berlin and his Kangaroo team as well as the *Hausdorff Center for Mathematics* in Bonn for the many occasions they gave us to develop material and to conduct workshops on different topics.