A vector is a line segment between two points? - Students’ concept definitions of a vector during the transition from school to university

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The vector concept is an important concept, with which students are confronted in their first year of university. To be able to build on the students’ previous knowledge it is important to find out what they know about vectors from school. This study aims to find out what university freshmen think a vector is. We therefore analyzed common German school textbooks to find out how the vector concept is introduced and what conception of a vector students might have. In addition, we administered a short pretest, in which students were asked what a vector is and to explain vector addition and its properties. We found out that freshmen state a lot of individual concept definitions. The majority stated geometric ones, which were mostly inadequate. Furthermore, the students’ explanations about vector operations were not based on the representations chosen in their definitions.

Keywords: Vectors, concept definition, transition, textbook analysis.

Introduction

Many students have problems during the transition from school mathematics to university mathematics. To reduce the students’ difficulties during this transition the Ministry of Innovation, Science and Research initiated the Studifinder project. Part of this project is the Studikurs which is an e-learning based bridging course for mathematics developed at University of Paderborn since 2014 (Colberg, Mai, Wilms, & Biehler, 2015) and is available since summer 2016. Students can use the course to fill gaps in mathematical knowledge from school and at the same time contents are presented on a more elaborated level than the school level with a focus on accurate language and notions as it is expected at university as a further means of preparation.

An important concept with which students are confronted at school and university (at a more abstract level) is the concept of vectors. In early 2016 we started the development of a chapter on vectors for the Studikurs by looking into school textbooks to get an inspiration how the concept could be introduced. While looking into many of these textbooks we observed the following:

1. The formal definition – that is either geometric as infinite sets of arrows with the same direction and length or symbolic as triple of real numbers – is often not used again in the chapters following the introduction of the vector concept.

2. There is often no distinction between the mathematical object “vector” that has been defined before and its many different representations.

These observations led us to the hypothesis that it is not clear for the students which of the introduced objects at school related to vectors really is a vector. This hypothesis was examined by
analyzing the common German school textbooks dealing with the vector concept and by administering a short pretest to university freshmen. The results are presented in this paper.

**Theoretical background**

Although every mathematical concept has a precise definition, students need to give it a meaning by operating with the concept (maybe just mentally) in order to understand it. Tall and Vinner (1981) use the term *concept image* to describe all associations students may have acquired by operating with it. These include examples, counterexamples, visualizations as well as properties of the concept. In order to specify the concept with words it has a *concept definition*. This can either be the formal definition accepted by the mathematical community, or the students’ reconstruction of a definition of the concept from their *concept image* (more precisely from the parts of the concept image that were activated during this reconstruction process, which Tall and Vinner (1981) call *evoked concept image*). In the latter case Tall and Vinner (1981) call it *personal concept definition*.

The formal definitions of the vector concept the students might have learned at school are the following ones, which were found out by analyzing German school textbooks:

1. A vector is an infinite set of arrows with equal direction and length. (Bigalke & Köhler, 2012; Bossek & Heinrich, 2007; Brandt & Reinelt, 2009; Weber & Zillmer, 2014).
2. A vector is a triple of real numbers or as a matrix with one row. (Alpers et al., 2003; Artmann & Törner, 1984; Griesel, Andreas, & Suhr, 2012; Griesel & Postel, 1990)

However, the students’ personal concept definitions, which they reconstruct from their concept images, may be different depending on their experiences with the concept.

In the following we present for each of the two the two formal definitions of the vector concept how they are introduced in German school textbooks and what possible personal concept definitions university freshmen might have assuming these introductions formed their concept image at school from which they reconstructed their personal concept definitions. Afterwards it is discussed how the formal definitions are used further in the books when operating with vectors and how this might again influence the students’ concept definitions reconstructed from their evoked concept image.

**Analysis of books using the geometric definition of a vector**

The geometric definition as infinite set of arrows with same length and direction is often motivated by translations (Bigalke & Köhler, 2012; Weber & Zillmer, 2014), and sometimes even defined by these, (Brandt & Reinelt, 2009). The translations are then represented by arrows with the same length and direction. Afterwards the students are told that these arrows all describe the same translation and can therefore be identified as the same object (see, e.g., Weber and Zillmer (2014)). That path would lead to the adequate concept definition of a *vector as infinite set of arrows with equal length and direction* (*D1*). However, this identification step is rather difficult as is denoted in literature, and may result in the inadequate conception that a *vector is considered as a single arrow* (*D2*) (Malle, 2005). The motivation of the formal geometric definition as set of arrows with equal length and direction may also lead the students to think that a vector is a translation. While a definition of a vector as translation map on the whole plane would be correct, literature shows that
translations are often understood as a motion of an object (Yanik, 2011). So the students might think of a *vector as a translation of an object or translation of a point* (D3). Yanik (2011) also found out that the connection between a vector and a translation is often not understood. Yanik (2011), e.g., found out that many teachers thought that a vector only gives the direction of a translation. This might lead to the misconception: *vector as direction indicator* (D4).

Besides using translations, some books also motivate the arrows in the space as a quantity characterized by length and direction in physical contexts like speed or force (Bossek & Heinrich, 2007; Weber & Zillmer, 2014). The recognition that two of these arrows can be considered as the same because only the magnitude and the direction matter (e.g., for the resulting movement of an object) leads to the adequate concept definition of a *free vector, which is a quantity characterized by length and direction and represented by a free movable arrow* (D5) (Watson, Spyrou, & Tall, 2003). But since forces are normally considered as dependent also on the position they operate on (Watson et al., 2003), this approach can again lead to the students to the consideration that *a vector is a single arrow* (D2).

The geometric definition requires not only a lot of effort for its introduction, but is also difficult to handle afterwards. In literature this is denoted as a lack of operability of the definition (Bills & Tall, 1998). The geometric definition is, e.g., difficult to handle when defining vector operations because for all operations the independence from the chosen representative of the vector has to be justified. In some books, this problem is discussed (Weber & Zillmer, 2014), others ignore it and vectors are just identified with arrows when defining vector operations geometrically (Bossek & Heinrich, 2007). This can lead again to the misconception of a *vector as a single arrow* (D2). Another option to deal with these difficulties is demonstrated in Bigalke and Köhler (2012). There the addition is defined via the addition of the components of the symbolic representation (just after its introduction) and from there on the geometric addition just serves as a visualization. This is at least inconsistent with the abstract concept of a vector as an element of a vector space, which consists of a set like equivalence classes of arrows with equal length and direction, and operations on this set satisfying certain axioms. So the students might not be able to embed their geometric concept of vectors into the formal theory of vector spaces properly.

After the introduction of the vector operations and their properties the definition as a set of arrows (or as a translation) is not used again (Bigalke & Köhler, 2012; Brandt & Reinelt, 2009; Weber & Zillmer, 2014) explicitly. Instead, in the following chapters dealing with analytical geometry, single arrows and their corresponding number triples are used to describe geometric objects. This can lead to a loose connection between the formal definition and students’ concept image from which they might deduce their own concept personal definition (Vinner, 2002). The resulting personal concept definitions in this case would be: *vector as a single arrow* (D1) or *vector as a number triple* (D6).

In summary, if the vector concept was introduced geometrically as infinite sets of arrows with equal length and direction, the following concept definitions can be expected: *vector as an infinite set of arrows with the same length and direction* (D1), *vector as a single arrow* (D2), *vector as a translation of an object or translation of a point* (D3), *vector as direction indicator* (D4), *vector as a quantity characterized by length and direction* (D5), or *vector as a triple of numbers* (D6). The
personal concept definitions \( D1 \) and \( D6 \) correspond directly to possible formal definitions of the vector concept, \( D5 \) is also an adequate conception, in which the equivalence of arrows with equal length and direction is realized by independence from the space, \( D2 \) and \( D3 \) are formally imperfect concept definitions (\( D2 \) does not take into account that vectors are equivalence classes, \( D3 \) does not take into account that a translation is a mapping on the whole plane) and \( D4 \) is a misconception.

**Analysis of books using the symbolic definition of a vector**

The symbolic definition is often motivated geometrically by translations or arrows (Alpers et al., 2003; Griesel et al., 2012) or as coordinates of the points in the space (Griesel & Postel, 1990). Sometimes it is introduced earlier in the theory of systems of linear equations (Artmann & Törner, 1984). The symbolic definition of a vector has the advantage that it allows a flexible interpretation as a point or an arrow. This can avoid the discussion about the equivalence of arrows (see, e.g., Alpers et al. (2003)). However, besides the already mentioned misconception of a vector as a single arrow, this flexibility can lead to another inadequate conception: vector as a point (\( D7 \)). The identification of vectors and points becomes problematic in higher mathematics in the theory of affine spaces, in these objects have to be considered as different (Henn & Filler, 2015).

The way the vector concept is introduced in Artmann and Törner (1984) can also lead to another adequate concept definition. Artmann and Törner (1984) restrict their visualizations of vectors on points and arrows from the origin. If students identify the number triples with these arrows from the origin they could consider a vector as an arrow from the origin (\( D8 \)). These arrows from the origin can really serve as objects, with which vectors and their operations can be defined.

Unlike the geometric definition of a vector as a set of arrows the symbolic definition is very operable when defining vector operations and the justifying their properties like the commutative law. However, some authors do see the necessity of talking about them (Alpers et al., 2003; Artmann & Törner, 1984; Griesel et al., 2012). The triviality of these properties in \( \mathbb{R}^n \) is also noted in literature (Harel, 2000). However, the symbolic definition can also be difficult to handle in the case of the definition of geometric concepts related to vectors that are like the norm of a vector. Algebraic definitions of these concepts like in Alpers et al. (2003) seem unnatural without further explanation. Geometric definitions of these on the other hand (like in Griesel et al. (2012)) require a careful distinction between the defined object vector and its representations as single arrows.

After the introduction of the vector operations, the concept is mainly used in geometrical settings (describing lines and planes in the space). This might cause that the students do not identify vectors with the object originally defined object ‘triple’ but with imperfect geometrical representations like points (\( D4 \)) or single arrows (\( D2 \)) (they might reconstruct their personal concept definitions of vectors from these imperfect representations and not from the formal symbolic definitions).

In summary, the symbolic approach can lead to two further personal concept definitions besides the intended definition of a vector as a triple (\( D6 \), which were not mentioned yet: vector as a point (\( D7 \)) or vector as an arrow from the origin (\( D8 \)). The identification of symbolic vectors with arrows from the origin is not problematic because the latter ones can really serve as objects, on which the vector operations can be defined. The identification of vectors with points, however, can cause conflicts later in the theory of affine spaces, in which these two objects have to be distinguished.
Methodology of the empirical study

Research question

What concept definition of the vector concept do students have when entering university?

Data Collection

In September 2016 a short pretest was administered to 103 university freshmen in a mathematics bridging course at the University of Paderborn. These students were either freshmen in mathematics major or in mathematics for teachers at grammar school. The pretest consisted of three open questions:

1. What is a vector?
2. Explain how you add two vectors \( \vec{a} \) and \( \vec{b} \).
3. Explain, why for all vectors \( \vec{a} \) and \( \vec{b} \) the following is valid: \( \vec{a} + \vec{b} = \vec{b} + \vec{a} \).

The first question was asked to find out what personal concept definition of a vector in the sense of Tall and Vinner (1981) the students have. We did not ask for a definition because we did not want the students to try to recall the formal definition they had learned at school from memory but to specify the concept with their own verbalizations (reconstructed from their concept image). We also did not use the term “definition” because we suspected that many students were not familiar with the term from school and might get confused with this term.

The other two questions were asked to further analyze if the students use the defined objects to explain vector operations and their properties. This is important for a proper embedding of the old vector concept into the abstract notion of a vector space, which is a set with operations defined on the same set. However, this problem will be investigated later.

Data Analysis

The answers to the first question “What is a vector?” were categorized by using possible personal concept definitions deduced from the analysis of the textbooks (see theoretical background, categories D1.,...,D8). Furthermore, four other categories have been added. The first one, a vector as an element of a vector space was added before the analysis because although this generalization is not taught at school, it may happen that some students heard about it (e.g., in mathematical clubs at school). The other categories contained inadequate personal concept definitions that often showed up during the analysis: A vector as a line segment, a vector as a line and a category containing other wrong concept definitions that had not been mentioned yet.

The whole category system of 12 categories is shown in Table 1. The first five categories are personal can be considered as adequate, which means that objects described in the definition can serve as concrete examples for vectors of a vector space (if suitable operations are defined between them) or if vectors are already considered as elements of vector spaces. Categories 6 and 7 contain imperfect or incomplete concept definitions, categories 8-12 contain inadequate concept image definitions, which can be considered as misconceptions.
Two of the authors coded the data from the questionnaire separately. The interrater-reliability coefficient Cohen’s Kappa was $\kappa=0.803$, which is good. Afterwards they discussed the answers they had coded differently and agreed on a categorization.

**Results of the study**

The answers of the students’ concept definitions of the vector concept are shown in Figure 2.

<table>
<thead>
<tr>
<th>Category</th>
<th>Prototypical Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Set of equivalent arrows</td>
<td>“An infinite set of arrows in the space, which are equal in their length and their direction.”</td>
</tr>
<tr>
<td>2. Quantity with direction and magnitude</td>
<td>“A vector gives information about the length and the direction of an element in an at least two-dimensional space.”</td>
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<tr>
<td>3. N-tuple</td>
<td>“A matrix with n rows and just one column.”</td>
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<tr>
<td>4. Arrow from the origin</td>
<td>“A vector describes a journey to a point of $\mathbb{R}^3$. ”</td>
</tr>
<tr>
<td>5. Element of a vector space</td>
<td>“An element of a vector space.”</td>
</tr>
<tr>
<td>6. Translation of a point or an object</td>
<td>“A vector is a translation of a point in the coordinate system to another point.”</td>
</tr>
<tr>
<td>7. Arrow or line segment with direction</td>
<td>1. “Visually: an arrow.”</td>
</tr>
<tr>
<td></td>
<td>2. “A vector is a line segment with direction.”</td>
</tr>
<tr>
<td>8. Direction</td>
<td>“A vector points from a point into a direction.”</td>
</tr>
<tr>
<td>9. Connection between points or line segment</td>
<td>1. “A connection between two points in the 3-dimensional space.”</td>
</tr>
<tr>
<td></td>
<td>2. “A line segment in the 3-dimensional space”</td>
</tr>
<tr>
<td>10. Point</td>
<td>“A vector describes a point in the coordinate system.”</td>
</tr>
<tr>
<td>11. Line</td>
<td>“A vector describes a line in $\mathbb{R}^3$. ”</td>
</tr>
<tr>
<td>12. Other wrong definition</td>
<td>“A multidimensional object, which is located in a multidimensional space.”</td>
</tr>
</tbody>
</table>

**Figure 1: Answer categories to the question “What is a vector?”**

The bars of adequate personal concept definition (which correspond roughly to possible formal concept definitions of the vector concept) are marked green, not fully adequate concept definitions are marked yellow, inadequate concept definitions are marked red. As can be seen in Figure 2 the
students had a variety of individual concept definitions of the vector concept when entering university. Most of them have a geometrical basis. However, in most cases these geometric concept definitions were either imperfect (the yellow bars, in which either the nature of a vector being an equivalence class was not mentioned or in which a vector was considered as a translation of points or objects and not as translations of the whole space) or inadequate (the red bars). Nevertheless, even the inadequate personal concept definitions also generate concept images which contain some correct properties and associations of the vector concept (e.g. if a vector is considered as a line segment, it is considered as an object “of finite length”).

**Conclusion and outlook on possible further research**

Our study shows that the students have a variety of concept definitions of what a vector is when entering university. The majority stated geometric definitions which were mostly inadequate definitions in the sense that they cannot be properly formalized or embedded into the common definition of a vector from mathematics. This indicates that it is difficult, not only for the students, to fully grasp these approaches to the concept of vector. However, many students seem to be familiar with the symbolic definition of a vector as an n-tuple and it can be interpreted in a manifold of representations. This property of the n-tuple approach seemed very appealing to us and it inherently avoids the obstacles of a mathematically adequate geometrical approach towards the definition of a vector. Particularly the rigorous dealing with equivalence classes and the independence from the chosen representatives can be avoided with this approach. After the analysis of the answers we now have an overview what concept definitions we can expect the students to bring into class. It might be advisable to confront the students with the most common misconceptions we found and to keep in mind that freshmen do not come to university with a common idea on what a vector is.

For further research we will look into the students’ answers to question 2 and 3 more thoroughly with a respectively elaborated theoretical framework. A first look into the data suggests that students use vector representations to explain vector addition rather than relying on the operational part included in the vector concept. We plan to do a follow up study to take a look what influence the linear algebra lecture had on the students at the end of the currently ongoing winter term 2016/17. To investigate this a similar questionnaire will be deployed but we will carefully reconsider how we phrase the questions, e.g., while it seemed unsuitable to ask for a definition at the beginning of their studies at university it seems legit to do so after their first term. Especially interesting will be whether addition and commutativity will be explained as inherent to the vector concept or if students still rely on representatives to explain addition and commutativity.

**References**


