Primary source projects in an undergraduate mathematics classroom: A pilot case in a topology course

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Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) is a five-year, seven-institution collaborative project funded by the US National Science Foundation to design, develop, test and evaluate curricular materials for teaching standard topics in the undergraduate mathematics curriculum in the United States via the use of primary historical sources. Three short projects designed for use in a topology course are described, together with elements of a pilot study that collected data from students in the course to evaluate changes in attitudes toward mathematics and its study.

Keywords: Instructional materials, primary historical sources, topology, worldviews.

Introduction.

There are numerous motivations and benefits for incorporating the use of primary sources into undergraduate mathematics teaching. Primary among the cited motivations is that providing students experience with reading texts in which the genetic development of a topic is presented gives them an opportunity to expand their mathematical education in such a way that includes both traditional and modern methods of the discipline (Fried, 2014; Laubenbacher et al., 1994). Another motivation is that using original sources in the teaching of mathematics makes it possible to contextualize the mathematics in ways that many textbook treatments do not afford. That is, original sources place particular mathematical ideas in the context and setting of the investigations in which the author was engaged at the time. As a result, the problems with which the author was struggling, and the motivations for solving them, are often more clearly and naturally described, and more compelling than traditional textbook expositions. Exposing the original motivations behind the development of esoteric mathematical concepts may be especially critical for placing the subject “within the larger mathematical world,” in the hope of making it more accessible to students (Scoville, 2012). Furthermore, primary texts seldom contain the specialized vocabulary that comes with later formalism, promoting access to the ideas by students with a wider range of backgrounds than is achieved with more standard presentations.

Many mathematics instructors interested in bringing the history of mathematics to the classroom question the use of primary historical sources in light of the increased availability of high quality secondary historical sources (e.g., Katz, 2009). Such resources may suffice to help students reap some of the benefits of original works; however, they carry their own difficulties. For example, there is a risk of placing too much emphasis on the history of mathematics per se, as opposed to using that history to support the learning of mathematics. Other key differences between using primary and secondary sources are described by Jankvist (2009):

> When using secondary sources, the students are exposed to a given historian's presentation and, possibly, interpretation of history, and they must make their choices based on this (Furinghetti, 2007, p. 136). When reading original sources, the students must, on the other hand, perform their own interpretation of what actually took place, why a certain mathematician developed a theory in one way or another, whether or not what is written is true, what internal and/or external forces drove the development of the work, etc. … The extent to which original sources are being used does, of course, have an impact on what the students learn: what students may gain from just "sniffing" at a few picks from an original source and what they might learn from being exposed to systematic readings of original sources are immensely different things. (p. 250)

In this final statement, regarding "what students may gain from just 'sniffing' at" selections from an original source, Jankvist points to the primary focus of the *Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources* (TRIUMPHS) project.

The TRIUMPHS project.

In 2015, a seven-institution collaborative project to design, test, and evaluate curricular materials for teaching standard topics in the university mathematics curriculum in the United States via the use of primary historical sources was funded by the National Science Foundation. The TRIUMPHS project seeks to help students learn and develop a deeper interest in, and appreciation and understanding of, these mathematical concepts by crafting educational materials in the form of Primary Source Projects (PSPs) based on original historical sources written by mathematicians involved in the discovery and development of the topics being studied. These PSPs contain excerpts from one or several historical sources, a discussion of the mathematical significance of each selection, and student exercises designed to illuminate the mathematical concepts that form the focus of the sources. PSPs are meant to guide students in their explorations of these original texts in order to promote their own understanding of those ideas. TRIUMPHS plans to work with mathematics faculty and graduate students from over forty institutions in the United States who will participate in the development and testing process of these PSPs. As part of this five-year project, impacts of the materials and approaches to implementing them will be investigated in terms of teaching, student learning, and departmental and institutional change.

Organization of the paper.

The remainder of the paper is organized into four sections. First, we describe the three "mini-PSPs" implemented by the third author during his topology course during the spring of 2016. Next, we present a broad overview of one of the components of research we are conducting during the five-year project, and then discuss a small subset of the data collected from five student participants. The paper concludes with a discussion of the implications from this small data set from the pilot year of the project, and a review of next directions planned for the research.

Primary source projects in topology: The case of three mini-projects.

The third author developed and implemented three "mini-PSPs" during the Spring 2016 semester at Ursinus College, a small liberal arts school outside Philadelphia, Pennsylvania. His experience using primary source materials in the classroom began in 2012 when he introduced them into a discrete mathematics course. His current interest in using primary sources in the classroom involves teaching topology. Three "mini-PSPs", each taking up two 50-minute class periods, were written for use in an Introduction to Point-Set Topology course, an upper-division elective course for mathematics majors and a designated "capstone" course intended to provide an experience for mathematics students to test and apply previously acquired mathematical knowledge and skills. The author taught this course twice before in a standard lecture-based style. A short description of each of the three projects is provided below.

The course was introduced with a mini-PSP titled "Topology from Analysis" that investigates a paper by Georg Cantor (1872) in which he considered a problem in Fourier series, namely, if a function has a Fourier series expansion, when is such an expansion uniquely determined? Cantor proved in a previous paper that two convergent trigonometric series with equal sums have the same coefficients (uniqueness theorem), even if – for a finite number of values of the variable – the series either fail to converge, or converge toward different sums. Could this theorem be extended to certain infinite sets of points? After reading Cantor's statement of the problem, the students explore simple examples of infinite sets where such an extension is possible. They investigate what properties these examples have that allow for such an extension. The desirable properties ultimately prompted Cantor to define concepts like limit points, derived sets, point sets, and iterated derived sets. The students were then able to use these concepts to prove a more general theorem. Even though these concepts were used to prove a result in analysis concerning Fourier series, they are naturally topological concepts. Hence the project helped to connect analysis and topology, thereby motivating new definitions through the need to clarify concepts, rather than introducing them as standard jargon.

The second mini-PSP focused on the topological concept of connectedness. It again considers a work of Cantor (1883) and his study of the continuum. Students follow Cantor's musings concerning the best way to define such a concept. After he defined a perfect point set based on derived sets, he investigated whether the property of perfectness is sufficient to characterize a continuum. The students wrestle with this question, and eventually observe that such a definition will not suffice. Cantor then proposed an additional property, connectedness, which he defined using a metric. After reading Cantor's definition of connectedness, students examine a work of Jordan (1883, pp. 24-27) which viewed connectedness in terms of separation of point sets into components. This introduces the students to a new conceptual perspective for the same notion considered by Cantor. Next, students read from a paper by Schoenflies (1904, pp. 209) in which it was proven that connectedness is a topological invariant. To do this, Schoenflies required a definition of connectedness that does not appeal to a metric, and is therefore purely topological. Finally, students read from a work of Lennes (1911, pp. 287, 303) in which he attempted to give a proof of the Jordan Curve Theorem and gave yet another definition of connectedness. Students are then asked to show that Lennes' definition is equivalent to the definition used today.

The final mini-PSP used in the course studies excerpts from a paper of John Henry Smith (1874) on discontinuous but integrable functions. Smith intended to provide a counterexample to a "theorem" of Henkel. He constructed an integrable function that is "very badly" discontinuous. Students are led through Smith's work involving the definition of the concept of nowhere dense set in order to construct such functions. As in the first mini-PSP, students are exposed to a problem that motivates the need for a new mathematical concept by abstracting the essence of particular examples in order to capture the essential properties that a set must satisfy to prove a result. Ultimately, this project culminates in Smith's construction of a generalized Cantor set. A function that is continuous except on a generalized Cantor set is then seen to provide a counterexample to Henkel's claim.

A major benefit of these mini-PSPs is that they naturally induce sophisticated discussions about the mathematics by students. During classroom implementation, students were observed to be carefully and thoughtfully working to understand concepts, answer questions, and pose their own questions and conjectures. One such instance occurred when students began to question Schoenflies' definition of connectedness, without prompting from the instructor. The question was raised as to what Schoenflies meant by the phrase "... can be decomposed ..." One student suggested that he meant a partition, but it was soon realized that such an interpretation would be too general. Others then began to modify the partition idea to make it work. This sort of high-level engagement had not occurred in any previous topology course taught by the instructor.

Description of the research.

The TRIUMPHS project includes an evaluation-with-research (EwR) study, designed to provide both formative and summative evaluation of the key project activities. In the original EwR study design, we designated three project components for which we would conduct extensive research and evaluation, and designated these as "student change," "faculty expertise," and "development cycle." Here we only describe the relevant components of the "student change" aspect of the EwR study.

Rationale and research questions.

For decades much of the research literature on the impact of the history of mathematics on students, particularly at the secondary level or post-secondary level, was focused on students' attitudes (e.g., Marshall, 2000; McBride & Rollins, 1977). There was scant focus on the use of primary sources as a classroom tool in the early work in the field of history in mathematics education. However, more recent work on the use of primary sources has been done in countries such as Denmark (e.g., Kjeldsen & Blomhøj, 2012) and Brazil (e.g., Bernardes & Roque, 2016), while such research has not yet been conducted with student populations in the United States. Thus, we are committed to investigating the ways in which mathematics students respond to concepts within the undergraduate curriculum that are taught via primary sources. To this end, we have developed the following research questions:

1. As a result of engaging with PSPs, what changes do students report in their attitudes and beliefs about learning mathematics?
2. As a result of engaging with PSPs, do students report any change in their mathematical worldview, and if so, what is the predominant view change?
3. In what ways does the use of PSPs influence mathematics (or mathematics-related major) students' beliefs and perceptions about mathematics?
4. As a result of engaging with PSPs, what do students report as challenges and benefits of learning from primary sources?

5. What is the potential of a given PSP (specifically the original source material within the PSP) to promote the learning of meta-discursive rules in mathematics?
6. What is the potential of a given PSP to promote student reflection on meta-rules in conception formation?

Pilot study: The case of a topology course.

It is important to note that only the first four research questions listed above had been identified in this pilot year, as the TRIUMPHS team was still developing instruments and refining the questions that were formulated in the original grant proposal. Also, the only data sources available for analysis were student pre- and post-course surveys, student work samples (from the three mini-PSPs), and instructor surveys and post-implementation reports. Data were collected from four undergraduate mathematics courses during the pilot year of the TRIUMPHS project: two courses in Fall 2015 (geometry; analysis) and two in Spring 2016 (abstract algebra; topology). In the topology course, the three mini-PSPs described above were implemented and tested for the first time by the third author. In this paper we report data that inform the first, second, and fourth research questions listed above for the five of eight students enrolled in that course who consented to participate in the research and for whom we obtained a complete set of data.

Exploring student responses: Research questions 1 and 4.

Our initial pre- and post-course surveys (to which students responded before instruction on any PSP occurred and again at the end of the course) only contained one pair of open-ended questions asking students to identify what they enjoyed most and least about studying mathematics:

> What do you enjoy most about studying mathematics. (Explain briefly.)
>
> What do you enjoy least about studying mathematics. (Explain briefly.)

For this small group of students, pre- and post-course responses were mostly stable for this pair of questions. This could be a function of the fact that all five students were in the final two years of their undergraduate programs. However, we discuss two interesting pre-/post-course survey pairs of responses below.

First, in response to the second prompt, Student 2 stated on the pre-course survey that she was “not fond of professors/texts that state a definition/theorem/idea without giving any hint to how that conclusion was derived, either historically or through proof/explanation.” By the end of the course, however, her response no longer referred to lacking historical grounding or a thorough proof or explanation. Instead, her response focused on the fact that “it is very easy to go through math classes and fall behind. If you don’t understand one topic, the class often moves on without you…” (post-course survey).

A similar change in identifying what he enjoyed least about studying mathematics occurred with Student 3. On the pre-course survey, Student 3 stated that he least enjoyed “how everything is given to you but never comes with an explanation of where the math is coming from.” However, in response to the same item on the post-course survey, Student 3 only commented that he disliked having to remember definitions and equations. It is possible that for Students 2 and 3 that the historical sources related directly to the formal content of the course satisfied their initial “sore spot” with regard to what they previously enjoyed least about studying mathematics. Since there was no effort in the pilot year to ask for student clarification (e.g., via follow up interviews), we cannot link the change in the sample responses presented here as resulting from students' engagement with the PSPs. However, the responses signal potential interesting outcomes and we have modified our pre- and post-course surveys and have added post-PSP surveys to further investigate this potential influence.

We also asked students to describe their experience using the mini-PSPs in the topology course, as a way to explore the benefits and obstacles they identified. The student responses were encouraging, and in important ways their responses point to the underlying effect of engaging with materials that provide an opportunity to understand the evolution of a mathematical concept. Here, we provide a sample of three of the students' descriptions:

> Student 1 (Mathematics major): "We used these sources to learn about topics such as connectedness as we thought about origins of the idea and read about how the definitions changed the longer it was studied."

> Student 2 (Computer Science major): "Each PSP was an interesting addition to class, and it was unique to be able to see the process of mathematical ideals through the eyes of the various mathematicians."

> Student 5 (Physics & Mathematics major): "I enjoyed the document on definitions of connectedness, because it showed how mathematics is really done and how it takes time to accurately articulate an idea. I also liked the first document because it helped motivate topological ideas and illustrate the connection between topology and analysis."

Exploring student responses: Research question 2.

To address the second research question, we included a subset of 20 items from Törner (1998) on the pre- and post-course surveys. Törner and his colleagues surveyed 310 German secondary mathematics teachers in order to identify attitudes about mathematics that captured "the essence of mathematics," where they defined "mathematics as a field and not as a subject taught in school" (p. 119). In doing so, they identified attitudes relating to four aspects of a mathematical worldview: scheme, formalism, process, and application. Tables 1 and 2 report the pre- and post-survey means of students' responses for these items. Item responses ranged from 1 to 5, where higher values indicate stronger association with that particular worldview. The predominant worldview of the Spring 2016 topology students on both the pre- and post-course survey was the process view.

Mathematical worldview	Student 1 (female; 4th year[1])	Student 2 (female; 4th year)	Student 3 (male; 3rd year)	Student 4 (male; 3rd year)	Student 5 (male; 4th year)
Scheme	3.4	3.2	3.4	**4.2**	1.6
Formal	3.8	3.4	3.8	4	4.4
Process	**4**[2]	**4.6**	**5**	3.6	**4.8**

[1] In the US college and university system, typical undergraduate degrees are four years in duration.

[2] The highest worldview score per student (pre- and post-survey) is identified in bold.

Application	3.6	3.4	3.6	3.4	3.4

Table 1: Mathematical worldview of Topology students, Spring 2016 sample (pre-survey)

Mathematical worldview	Student 1	Student 2	Student 3	Student 4	Student 5
Scheme	3.4	2.6	4.2	3.6	1.4
Formal	**4.8**	3.4	4	3.8	4
Process	4.2	**4.4**	**5**	**4**	**4.4**
Application	3.8	3.4	4.4	3.8	3.4

Table 2: Mathematical worldview of Topology students, Spring 2016 sample (post-survey)

Törner (1998) reported on the relations among these four aspects, stating, "the formalism and scheme scale represent both aspects of the static view of mathematics as a system and intercorrelate highly" (p. 125). However, these two aspects of a static paradigm "correlate with the process scale in a significantly negative way" (p. 125), a finding that appears to hold for several students in our sample. For example, where there are lower means on the scheme and formal aspects (e.g., Student 2 pre/post; Student 3 post; Student 4 post), higher mean values occurred on the process aspect.

Discussion.

This paper highlights the promise of robust investigations and implications that may result from the EwR efforts of the TRIUMPHS project. The project's pilot year enabled us to trial student and instructor survey instruments and data collection procedures. We chose the topology course as a case because of the particular nature of the PSPs ("mini" as opposed to full-length), the students' maturity (juniors and seniors), and the expectation that many of the courses participating in TRIUMPHS will also likely have small enrollments. As a result of the pilot year, we have significantly modified our survey instruments and research questions, and our aggregate student population has now increased. As we move forward, our research plans include conducting multiple analyses to identify trends in students' mathematical worldviews (within courses, across courses, and disaggregated by other demographic characteristics). We have also developed and incorporated post-PSP surveys and are currently developing and piloting different interview protocols.

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