Principles of redesigning an e-task based on a paper-and-pencil task: The case of parametric functions

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Paper assessment tasks are often redesigned to function as digital assessment tasks. The research and design literature (Pead, 2010; Burkhardt & Pead, 2003) has reported on the challenges of such a transformation. We report on a study exploring the design principles of an e-assessment task, originally designed as a paper-and-pencil task and converted into an interactive diagram. We describe a paper task in the content area of parametric functions, and report on results from an experiment conducted with 39 high school students, who dealt with an e-task based on a paper task. Analyzing the results, we demonstrate that in a redesigned e-task based on a paper-and-pencil task, technology should allow self-reflection, promote learning during assessment, and guide the students to focus on the important details without unnecessary distractions.

Keywords: design, e-task, parameter, assessment

Goals and Theoretical Framework

Assessment sets priorities for learning and teaching: the results of assessment define what is taught and how it is learned. Summative assessment is known as "assessment of learning," whereas formative assessment is known as "assessment for learning," which involves weaving assessments directly into the fabric of the classroom or curriculum. We adopt the general term “assessment” used by Black and Wiliam (1998) to refer to all the activities undertaken by teachers that provide information to be used as feedback to modify teaching and learning activities. Because assessment is linked to learning, it is difficult to separate e-tasks designed for assessment from those designed for learning. Our research[1] focuses on design principles of e-tasks for assessment purposes, and therefore we assume that the students have already learned the material in class and should be working individually on the e-task, without the need of teacher mediation or other external help. It is assumed that students have not seen the tasks before, and therefore cannot solve them instrumentally, using a known procedure, but must work on them conceptually (Skemp, 1976).

In this paper, we explore design principles of an e-task that encourages exploration, based on a paper-and-pencil (P&P) task, in the area of parametric functions, which is central in algebra and is adequate for enhancing the abstraction of concrete situations (Drijvers, 2001). Solving parametric equations is different and more challenging than solving numerical algebraic equations, which are solved for an unknown that is a number. Naturally, when designing an e-task we should not translate from the paper but rather use successful principles of learning within the interactive environment to design the assessment tasks. Research shows that many complex issues arise when

[1] This study is part of the doctoral dissertation (to be submitted) of Galit Nagari Haddif, under the supervision of Prof. Michal Yerushalmy.

transferring paper-and-pencil tasks to computers. For example, if students are not familiar with the tools, the online environment may be a potential source of an additional "cognitive load" (Pead, 2010). Interactivity can spoil some tasks: for example, by allowing students to check all their answers, or by encouraging them to persist in trial-and-error experimentation, rather than engaging in analysis (Burkhardt & Pead, 2003; Nagari Haddif & Yerushalmy, 2015). Although the transition from a paper-and-pencil task to an e-task is not trivial, there may be an added value in the use of technology. For example, multiple linked representations (MLRs) both support and require tasks that involve decision making and other problem-solving skills, such as estimation, selecting a representation, and mapping the changes across representations (e.g., Yerushalmy, 2006).

With the Cabri software, Healy (2000) introduced soft and robust construction and found that despite the intention to encourage students to build robust constructions, in practice, some students preferred to investigate a second type of Cabri-object, soft constructions, in which one of the chosen properties is deliberately constructed by eye in an empirical manner, under the control of the student. Laborde (2005) referred to soft constructions as the "private" side of the student’s work, which is part of the solving process and serves as a scaffold to a definite robust construction. We suggest using soft constructions as a way of exploring and identifying dependences between properties, and as a gateway to a definite robust construction from a purely visual solution. Below we describe a task (Figure 1) taken from Taylor (1992, p. 204), and the reasons for which we decided to redesign it and convert it to an e-task. In general, Taylor’s rationale for this kind of task is to have a marked difference between being able to see ("sense") the solution geometrically and the ability to solve it algebraically. Interactive MLR technology offers dynamic interactions that can support the generalization of a graph into a family, offering sensuous support for finding an abstract parametric solution. This gap between interaction and abstraction is one of the challenges of assessment with interactive MLR technology. The following description of the e-task design and of the experiment conducted with the students who worked with this e-task, demonstrates some basic considerations and principles of designing an e-task based on a P&P task.

The original P&P task and its possible correct solutions

Research on mathematicians’ conjecturing and proving activity suggests that use of examples plays a critical role both in the development of conjectures and in their exploration, as well as in the subsequent construction of proofs of these conjectures (Lockwood, Ellis, & Lynch, 2016). Therefore, when dealing with the task (Figure 1), we can expect work that would look like Figure 2 (a): the students would sketch for themselves some exemplary lines through the origin.

At the right is the graph of the cubic equation $y = x(x-1)(x-3)$. Consider the family of non-vertical lines through the origin. How many intersections does each line have with the curve? (I) Begin by making a conjecture based on the picture. (II) Describe the family of lines algebraically, and verify your conjecture.

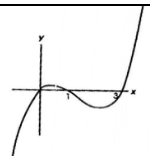

Figure 1: The original task as it appears in Taylor's book (Taylor, p. 204)

In this case, it may be difficult to make a generalization and diagnose the three different numbers of common points: one, two, and three common points between the family of lines $y = mx$ and the given function $y = x(x-1)(x-3)$. Moreover, one could start to investigate algebraically the mutual relationship between the two functions required in part (II) (Figure 1), and skip part (I).

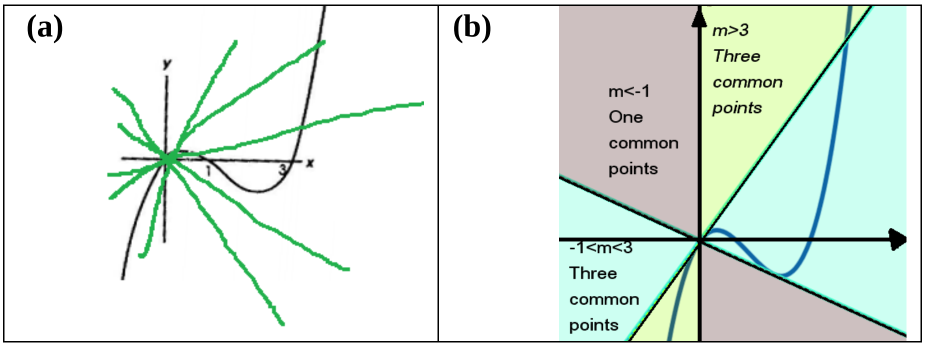

Figure 2: (a) Typical free-hand sketching used to conjecture about the intersections; (b) The domains and values of parameter m for all cases of number of common points

The abstraction and generalization needed to find and define algebraically the domains of the parameter m for each case number of common points (Figure 2 (b)) is a challenge. Generally, solving the case of two common points requires using two approaches, as shown in Figure 3.

The algebraic approach	$x^3 - 4x^2 + 3x = mx \Rightarrow x^3 - 4x^2 + 3x - mx = 0 \Rightarrow x\left[x^2 - 4x + (3-m)\right] = 0$ $x = 0$ is one intersection point. Therefore, when the equation $x^2 - 4x + (3-m) = 0$ has single solution: $\Delta = b^2 - 4ac = 16 - 4(3-m) = 0 \Rightarrow m = -1$ The other case is when the equation $x^2 - 4x + (3-m) = 0$ has two solutions, one of which is $x = 0 \Rightarrow 0^2 - 4 \cdot 0 + (3-m) = 0 \Rightarrow m = 3$
The calculus approach	$m = f'(0) \Rightarrow f(x) = x^3 - 4x^2 + 3x \Rightarrow f'(x) = 3x^2 - 12x + 3 \Rightarrow f'(0) = 3 \Rightarrow m = 3$ In the other case, let $(t, t(t-1)(t-3))$ be the point of tangency $m = \frac{\Delta f}{\Delta t} = \frac{t(t-1)(t-3) - 0}{t - 0} = (t-1)(t-3)$ $f(t) = t(t^2 - 4t + 3) = t^3 - 4t^2 + 3t \Rightarrow m = f'(t) = 3t^2 - 8t + 3$ $m = (t-1)(t-3) = 3t^2 - 8t + 3 \Rightarrow t^2 - 4t + 3 = 3t^2 - 8t + 3 \Rightarrow t = 2{,}0 \Rightarrow m_{1,2} = 3{,}-1$

Figure 3: The two main ways of solving the case of two common points

In both the calculus and the algebraic approaches, there is an "easy" value of m and one that is less obvious. When grappling with this task, students should "see" (and be able to calculate) that there are two possible values for parameter *m*, for which both functions have two common points: $m = 3$ or $m = -1$. Some students (see also Ron's thinking-aloud process in Figure 6) may skip one approach that reveals one of the values of m and move on to the other approach to obtain another value. A mathematical pedagogical discussion may address the manner in which the two approaches meet. This rich task concerns various mathematical concepts besides parametric functions: intersection points, tangency to a function, and mutual relationships between functions. It encourages making conjectures and aims to assess skills such as exploration, algebraic manipulation, and generalization of particular cases and examples.

Study

The methodology we used in our study is design-based implementation research (DBIR): it concerns design principles of innovative assessment items, which are best studied by iterative cycles of design. We redesigned the task (Figure 1) and converted it to an e-task. We conducted an experiment with 39 10th- and 11th -grade students who worked with the e-task. The students studied the standard curriculum with different teachers in the same school, without special emphasis on technology. Ron, one of the students, was thinking aloud during the solving process (Figure 6). The video recording of his thinking aloud enabled us to follow the process of task completion as it was taking place, rather than consider only its final product, and to listen to the problem-solving process.

Design considerations and possible correct solutions: Parts A and B

MLR experimentation first. In part A (Figure 4), the students used a dynamic applet that displays the function $f(x)$ and the parametric family $y = mx$ on the same coordinate system.

The following interactive diagram describes the functions $f(x) = x(x-1)(x-3)$ and $y = mx$. By dragging the red point, you can create different examples of mutual relationship between these functions.
Part A: How many common points are there for both functions? Submit three different screenshots, each one representing a different number of common points.**Part B**: For which values of m does the functions have one common point? Two common points? Three common points? Indicate all the possible values.

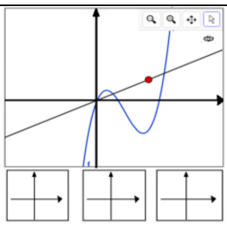

Figure 4: The designed e-task: parts A and B

Students became familiar with the activity and the givens, and were asked to submit three screenshots of three different cases of numbers of common points, in other words, three different "soft constructions" (e.g., Healy, 2000; Laborde, 2005), designed to support their generalization process and symbolic work required in part B. In the case of the parametric family in the MLR environment, any change in the value of m changes simultaneously the graphic representation of the relevant line. In designing this part, we wanted to make sure that students experimented with the applet, understood all the details and givens, and were exposed to many examples of the parametric function, so that in part B they could concentrate on the exploration activity with as little cognitive load as possible. An example of a correct solution is shown in Figure 5. The student submitted three different mutual relationships of the two functions, each with a different number of common points.

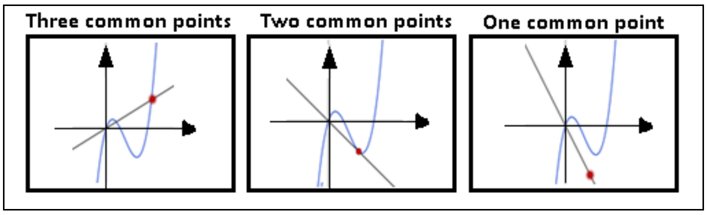

Figure 5: Example of a correct solution for part A

Minimal necessary tool set. There are deliberately very few tools available to the student: zoom in, zoom out, and move the coordinate system. The tools are designed to enable users to sense the task qualitatively, allow them to focus on the relevant parts of the graphing picture, and not to provide numeric information. This minimal design coveys the message that other parts of the tasks require numeric and symbolic calculation not provided by the interactive diagram. To summarize, the goals of part A are: (a) encourage student experimentation with the dynamic applet: feel/sense the givens and avoid cognitive load; (b) expose students to a variety of examples demonstrating the mutual relationship of the two functions; (c) engineer an experience-based conceptualization for solving the general case required in part B; and (d) assess the student's understanding of the givens and of what is expected of them (as well as some technical issues). Solutions for each number of common points appear in Figure 2 (b): one $m < -1$; two $m = 3$ or $m = -1$; three: $-1 < m < 3$ or $m > 3$.

Findings and data analysis: Parts A and B

Part A: Most students submitted correct answers for part A (Table 1). This is not surprising, because the purpose of this part was to encourage students to "sense" the problem and its givens. But 15% of students missed one case (of one or two common points). Possible reasons for this are that students were not experienced enough with the applet, that they did not use the tools to view all cases, etc. As mentioned above (Figure 3), it is easy to reach one of the two possible values of m, but while experimenting with the interactive diagram one may notice that there is another possible value of m. In Figure 6 we describe Ron's thinking aloud about a solution for part B.

Table 1: Submission characteristics of correct and incorrect answers to part A

Correct answers	Correct answer	29 (74.3%)
Incorrect answers	One missing case (of one or two common points)	6 (15.8%)
	Technical problem	3 (7.7%)
Not submitted		1 (2.6%)
Total		39 (100%)

In the beginning, Ron solved this part algebraically and found that m=-1 is the case in which the functions have two common points. By zooming in and out, he found that there is another value for m. The interactive diagram allowed Ron to connect between the algebraic and graphic approaches (Figure 6, line 9). Through experimentation, he tried to find the other value of m (Figure 6, lines 4-9). This demonstrates the power of technology as a tool that provides students means to reflect their solution, allowing learning to take place during a test. In practice, during the experiment some students noticed the missing value of m and tried to find it (not always successfully), either by expanding the solution using the same approach, or by changing the approach from algebraic to calculus or *vice versa*, as described in Figure 3.

Part B (Table 2): 18 of the solutions included one or both correct values for m (m=-1, m=3).

Table 2: Submission characteristics of correct and incorrect answers to part B

	N=39 (100%)		
Number of common points	**One**	**Two**	**Three**
Correct answer	15 (38.5%)	2 (5.1%)	2 (5.1%)

Partial answer	0 (0%)	16 (41%)	7 (17.9%)
Incorrect answer	8 (20.5%)	6 (15.4%)	13 (33.3%)
Not submitted	16 (41%)	15 (38.5%)	17 (43.6%)

Only two students submitted a completely correct answer; 15 students did not submit an answer. Others submited other values, probably as a result of calculation errors or because they were guessing. Only two students submitted a correct answer for the case of three common points (the same students who submitted correct solutions for the case of two common points). This may imply that in addition to the difficulty of finding both "critical" values of the parameter m (m=-1, 3), it is also difficult to generalize and formulate symbolically the possible domains of parameter m for the cases of three common points.

	Ron's thinking aloud	**Ron's actions on and with the screen**
1.	This is equivalent to solving this equation. Right? Right. Zero is always one common point... then... Then I can divide simply by x. Then I investigate the quadratic equation:	$mx = x(x-1)(x-3)$ $m = (x-1)(x-3)$ $x^2 - 4x + 3 - m = 0$
2.	I check when it has two solutions, one solution, or zero solutions. These are two common points in my opinion.	He looks at the case that is close to m=3.
3.	No, these are three common points... There is a certain m... It has to be minus 1	This is the value that he got through the algebraic calculations.
4.	Then where did I go wrong?	Graphically, he sees that there is another positive m, but he got only m=-1.
5.	I need to check this equation.	Refers to $mx = x(x-1)(x-3)$.
6.	I always have one common point. Then I can simply divide by x. I have a neat equation. I have to see when it is equal to zero. We need to check when the discriminant is positive, negative, or zero. m must be different from zero to have two common points. Then m equals to -1. m=-1 for two solutions.	He checks again his calculations: $mx = x(x-1)(x-3)$ $m = (x-1)(x-3)$ $x^2 - 4x + 3 - m = 0$
7.	I need to zoom in.	Ron uses the zoom in and out buttons to explore and distinguish between different cases.
8.	I don't know what is my analytic mistake...	
9.	This is the tangent, the tangent.	He finds the graphic meaning of the case of two common points. Using calculus, Ron finds the other value of m.

Figure 6: Ron's thinking aloud while working on part B

Discussion

Our first conjecture, that a dynamic and interactive MLR environment supports the generalization of a graph into a family was only partially confirmed. The results of part B reveal the complexity of the concepts involved in the tasks. Results may suggest the presence of a permanent tension when designing mathematical e-tasks. On one hand, we want to design an e-task for exploration that can be automatically checked. Therefore we tend to give students the opportunity to explore without any hints and without leading them to the solution. On the other hand, the task may be too difficult, and we may have difficulty assessing the students' knowledge and mistakes. In retrospect, this exploration e-task was too difficult; we should have divided the e-task into more than two stages and ranked the sub-tasks: first concentrate on the case of two common points, and only later on other cases. We are currently considering a refined design of this e- task.

Below we describe some basic design principles we gleaned from the experiment described above. Other design principles are described in Yerushalmy, Nagari Haddif, and Olsher (under review), Nagari Haddif and Yerushalmy (under review)**. When re-designing an e-task based on a P&P task, technology should provide the following: (1) Allow self-reflection:** When solving a P&P task, we have few means to reflect on our solution, especially not instantaneously. Use of an interactive diagram in an MLR environment together with manual calculations helps students control their actions and reflect on them during assessment, and check whether they are right or wrong, without telling them directly what the correct solution is. **(2) Promote learning during assessment:** A good assessment task is a learning task. Using an interactive diagram with parametric functions in assessment allows seeing many instances of the same family. This is an opportunity to see the parameter serve as a "generator" of functions that belong to the same parametric family. Experiencing with the dynamic diagram also encourages students to make conjectures and conduct interactive exploration during assessment. As demonstrated in Ron's case, technology has the potential to create a cognitive conflict and thereby provide a learning opportunity during assessment (Figure 6, line 4). Ron tries and succeeds in solving the conflict between the manual solution and the graphic representation on the screen. Had he solved the original task, he may not have noticed the "conflict" between the graphic representation and the symbolic calculation. **(3) Guide students to focus on the important details, without unnecessary distractions:** Although it is tempting to use the varied capabilities of technology, these might distract the students and produce negative effects. Therefore, it is necessary to focus on the real needs of the students and redesign the e-task to make students concentrate on the important details, without unnecessary distractions. This also implies that students understand how to approach the question. The design must reflect in some way the purpose for which the tool was created (Yerushalmy, 1999), and the cognitive load must be reduced as much as possible. We demonstrated several ways of doing this: (a) designing the task in a way that students move in stages away from using sensory knowledge in soft constructions toward experimenting with the interactive diagram to produce robust constructions, abstraction, and generalization. (b) designing the environment and the required solution in a way that defers engagement in numeric and symbolic activity: for example, eliminating grids and the option to enter expressions conveys the message that conjecturing comes before computations. (c) determining the minimal necessary tool set that is familiar to the students and common to other e-tasks. In Yerushalmy et al. (ibid.) we described in detail the tool set needed for calculus e-tasks, which enables automatic checking of the students' submissions.

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