Children's performance on a mathematics task they were not taught to solve: A case study

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A teacher documents how a task was used to elicit children's knowledge of multiplication as a precursor to learning the long multiplication algorithm. Sources of data include samples of children's written work, and transcripts of two 30-minute lessons. Children discussed various solution strategies. The teacher used time between the two lessons to select and sequence representative strategies to share. Strategies used by the children were diverse showing a wide range of multiplication-related knowledge among the children ranging from knowledge of repeated addition to knowledge of place value and the additive decomposition of numbers to various applications of the distributive property.

Keywords: Long multiplication, problem solving, third grade, mathematics laboratory.

Introduction.

A common pattern of mathematics teaching in some countries is where teachers present a problem to children, demonstrate how to solve it, and then set similar problems for the children to solve applying the strategy demonstrated by the teacher (Lyons, Lynch, Close, Sheerin, & Boland, 2003; Stigler & Hiebert, 1999). An alternative approach is to use problems to teach new mathematics. Children are presented with a problem of a type they have not solved before and asked to attempt it using the mathematics they know; solutions and attempted solutions are discussed and shared among children (Lampert, 2001). This paper presents a case study of one teacher using a problem solving approach to introduce long multiplication to children who have completed third class.

Long multiplication is introduced to children in Ireland in fourth class after learning short multiplication in third class. The current case study describes and analyses work done over two days in a mathematics laboratory school which took place in July 2016 in Dublin, Ireland.

Two research questions are addressed in this paper. First, what knowledge did children draw on to solve a long multiplication problem they had not been taught to solve? Second, what knowledge does the class possess that prepares the children for future work on long multiplication? These related questions will be answered drawing on data from the children's written work and transcripts of lessons over two days, each of about 30 minutes duration.

Theoretical Framework.

The theoretical framework for this study draws on two areas of research. The first is research on progressive schematisation or progressive mathematisation. This idea was inspired by the work of Freudenthal and Realistic Mathematics Education and refers to the ways children create and use their own algorithms to solve problems. These invented algorithms become increasingly sophisticated before students begin to understand and use the formal, more efficient algorithms (Treffers, 1987). Although other authors have written about invented algorithms (e.g. Kamii, Lewis, & Livingston, 1993), Treffers (1987) formalises the process and refers to vertical mathematisation (where algorithms are reorganized and refined) and to horizontal mathematisation (connecting the mathematics to real-life) (Selter, 1998).

The second area of research that frames this paper is research on the teaching of multiplication (Lampert, 1986). In particular she draws on four categories of knowledge which apply to how children learn mathematics. They are intuitive knowledge, computational knowledge, concrete knowledge and principled knowledge. Intuitive knowledge or naïve knowledge refers to how people in specific contexts invent ways to calculate in order to do their specific work; it may not transfer well to other contexts. Computational knowledge is the procedural knowledge that children typically use in school where standard algorithms are followed. Concrete knowledge is used when objects are manipulated to find answers. This may even extend to the use of rectangular grids which are sometimes used to compute numbers in multiplication problems. Finally, principled knowledge is knowledge that children can use, without necessarily understanding the meaning of what they are doing. Such knowledge might involve children drawing on principles such as the commutativity of addition or multiplication, the distributive property of multiplication over addition, or place value.

This framework will help to identify stages in the heterogeneous work of a class (Selter, 1998) of children as they attempt to solve problems for which they do not have a solution. It will also help to categorise the knowledge that children drew on in solving the problems.

Method.

Participants

Twenty-four children – 17 girls and 7 boys – were in the class which lasted for two hours per day over five days. The children had completed third class in ten different schools and therefore could be expected to be familiar with short multiplication but not to have done any work on long multiplication. The mathematics laboratory class was taught by the author and was observed by twenty-five teachers who were completing a summer course in mathematics.

All children at the summer school had completed third class in June and were entering fourth class in September. Although several topics were taught in the summer school, the focus of this paper is on the introduction of long multiplication. A word problem was chosen from Van de Walle: "The parade had 23 clowns. Each clown carried 18 balloons. How many balloons were there altogether?" (Van de Walle, 2001, p.182). Children worked on this problem collaboratively in pairs; they were encouraged to solve the problem and to be prepared to justify their solution.

Data Analysis

Two data sources form the basis of this study. The primary data are samples of children's written work and these are complemented by transcripts of dialogue from the summer school. Children completed their written work in squared exercise books using black pen. This was done to ensure that they would not erase work they were unhappy with or that contained errors. This rationale was shared with the children. All lessons were video recorded by two cameras – one focused on the children and one focused on the teacher. The videos were used to prepare lesson transcripts.

The research questions relate to the knowledge used by the children and available to the class as a resource for future learning. All children's written work completed in response to the problem was studied and compared to ensure that samples of every approach used were represented in the four samples of work which were selected for more detailed analysis. The chosen work samples were subsequently analysed to identify categories of knowledge that were evident in the work. The four categories outlined by Lampert (1986) were used to guide this analysis.

The transcripts were analysed for evidence of student mathematical knowledge. Although the categories identified by Lampert informed this analysis, the analysis was open (Corbin & Strauss, 2008) so references to knowledge not covered by the four categories could be identified.

Results.

Lesson plans

The multiplication problem was written out on a chart to be displayed in the class. It was planned that the children would work on it in pairs.

The following day the plan was to start the day by working on the same problem. Prior to this lesson, as the teacher I had had the opportunity to look at the work done by all the children and to identify and sequence four approaches that would be worth sharing with the entire class. The strategies selected involved repeated addition, repeated addition with some multiplication, multiplying using partial products and an attempt at the standard algorithm for long multiplication. The lesson plan refers to a pictorial representation of the problem that would help the children get an understanding of the dimensions of the problem. This was introduced in response to a similar approach used for various scenarios by Lampert (1986). The lesson plan concluded with the intention to ask the children to solve a different long multiplication independently.

Day 1

In the problem presented, two two-digit numbers needed to be multiplied. The calculation was embedded in a word problem referring to a setting to which most children in the culture can relate – many clowns each holding several balloons. The combination of relatively low two-digit numbers and the concrete image of the clowns with balloons would make it relatively easy for children to draw the scenario should they decide to do so.

The problem was written on a wall chart and an individual copy was given to each child. After a class discussion of the problem, children worked on it in pairs for seventeen minutes. The teacher circulated among the children monitoring the work of pairs. Teachers who were attending the summer course walked around the class observing the children working. Although they were asked not to interact with the children, on one or two occasions some did.

After the children had worked in pairs, the teacher asked one pair of children – Sandra and Lisa – to tell the class how they went about solving the problem. They had added eighteen and eighteen to make thirty-six. Then they added another eighteen. The teacher asked the class if this strategy were implemented properly, would it yield the correct answer, and based on their responses reminded them of the need to be systematic in recording their work.

A second pair of students, Chuck and Róisín, shared a different approach with the class. They wrote down twenty-three eighteens and multiplied twenty-three by eight and twenty-three by one. This would represent an understandable mistake where they forgot that the one digit in the eighteens represents ten rather than one.

At this stage the teacher adjourned the discussion and moved to another mathematical topic. Having concluded that part of the work for the day, the teacher could examine and reflect on the children's work in order to select and sequence material for discussion in the following day's lesson.

Day 2

Overnight the teacher looked at each child's work. No child had successfully used the long multiplication algorithm suggesting that, as expected, it had not been taught to any of the children prior to the summer school. Four examples were selected and sequenced in a way that was anticipated to tap into the children's current understanding, to show increasing efficiency or sophistication of solutions – progressive mathematisation – and to prepare the children for subsequent work on long multiplication. Although all children had worked in pairs, the teacher selected the work to be shared according to the clarity of the written work observed in individual children's copybooks.

First was Christine who had used a straightforward repeated addition approach. She had a pictorial representation of the problem with twenty-three faces and eighteen balloons over seven of them (see Figure 1). Next was Donal who had no pictorial representation but who also used a repeated addition approach. He had grouped ten eights where possible to multiply them and had multiplied twenty-three by ten (see Figure 2). Third was Fintan who solved the problem by calculating ten eighteens, another ten eighteens and three eighteens and then added the three calculations (see Figure 3). All three students had the correct answer of 414. The fourth student, Eileen, whose work was selected got the wrong answer but the strategy used seemed closer to the standard long multiplication algorithm. She wrote an account of what she did and of how she was thinking rather than just recording the calculation. She multiplied the three from twenty-three by the eight in eighteen and got twenty-four. She then multiplied twenty by ten to get two hundred. She refers to multiplying two by one and it is not clear if that is a precursor to multiplying twenty by ten (see Figure 4).

Despite the fact that Christine had set up the calculation to be solved using repeated addition (see Figure 1), she stated that to solve the problem she and her partner solved "drew a picture and we did loads of dots and we counted them all up." When challenged by the teacher about the repeated addition work in her copy, Christine responded that "I had a long sum but that didn't really work because I kept on losing count." This provided an opportunity to discuss with the class one problem that arises with repeated addition, and to prepare the class for seeking more efficient ways to calculate using long multiplication. The teacher did not ask Christine why it was easier to keep track of counting the balloons individually than adding 18 twenty-three times and that may have yielded information about a system she had developed to keep track of the balloons already counted.

Donal had a somewhat more sophisticated way of working with repeated addition (Figure 2). Although his layout of the problem looks similar to Christine's, he approached it as follows

(8x10) + (8x10) + (8x3) + (10x23)

In solving it this way Donal and his partner showed good understanding of the distributive property of multiplication. In their calculations Donal and his partner noted that the ones in the eighteens represented tens and not units. However, Donal's written recording of the work and his oral explanation of it suggests that they were not yet familiar with multiplying numbers by ten. In contrast, his classmate David stated that "When I'm multiplying by tens, I just add on another zero at the end of the number." Although the wording of "adding" another zero may not be helpful, he is referring the fact that multiplying a number by ten shifts each digit in the number one place to the left requiring a zero as a placeholder in the units place.

A more condensed understanding of the distributive property was apparent in Fintan's work (Figure 3). Unlike Donal or Christine he did not write out the calculation using repeated addition. Nor did he separate the tens and units in order to complete his calculation, which took the following form:

(18x10) + (18x10) + 18 + 18 + 18

Fintan sees his approach as being similar to Donal's and he states that "we basically did the same as Donal; we used hundreds, tens, and units." However, whereas Donal's approach was limited by apparently not being able to multiply two-digit numbers by ten, Fintan was able to make the calculation more efficient by multiplying eighteen by ten.

When asked to choose a preferred strategy from those presented by Charlotte and Donal or by Fintan. Four children preferred Fintan's approach on the basis that it is quicker and it requires less writing. Two claimed to prefer the repeated addition approach because it looked less complicated.

When children solved the first long multiplication problem, twelve of them used a variation of repeated addition, three used a form of the distributed property, three a variation of the conventional algorithm, the work of three children was unclear and one used counting. Following the discussion, seven children used repeated addition, four used a form of the distributive property, five used a form of the conventional algorithm, the work of six students was unclear and one used counting.

When the children were asked where the twenty-three (clowns) could be seen in Fintan's strategy, two children (Katherine, Ethna) found it difficult to identify. One, Doireann, successfully constructed an explanation with the teacher in the following exchange.

Doireann: So the eighteen times ten is done twice. So that would be like twenty there. And then…

Teacher: So, you're saying that this is ten clowns with eighteen balloons, and this is another ten clowns with eighteen balloons.

Doireann: Yeah.

Teacher: Is that what you're saying?

Doireann: Yeah.

Teacher: Okay. And what then?

Doireann: And then if you add that together that's twenty…

Teacher: Twenty clowns with eighteen balloons.

Doireann: Yeah, and down the bottom there, it's three eighteens. Add them onto the twenty and it's twenty-three.

Finally, Eileen was asked to share her approach. Initially she stated that she and her partner got the wrong answer. After reassurance from the teacher that the class could learn from the wrong answer, she shared her approach. She writes that "in her head" she laid out the problem as it would be laid out in the conventional algorithm for long multiplication. She multiplied the three units by the eight units and got twenty-four. Then she multiplied the two tens (of twenty-three) by the one ten (of eighteen). She added them together and got 224. Although the teacher sequenced Eileen's strategy after Fintan, it is conceivable that her understanding is more naïve than his because she may have been attempting to apply the algorithm for short multiplication to long multiplication without really understanding the distributive property. Nevertheless, it provided an opportunity for the teacher to introduce an illustration of the distributive property of multiplication to all children.

What Eileen had failed to do was to multiply the eight by twenty and the ten by three. In order to help her and her classmates visualize this, I presented a drawing of the scenario based on Lampert (1986). In this drawing (see Figure 5) twenty clowns were on a bus travelling to the parade and three clowns had to walk because there were only twenty seats on the bus. Twenty strings with balloons on them could be seen emerging from the bus, each string with one group of ten and one group of eight and the three clowns outside the bus held similar strings of balloons.

The class was asked how they could calculate the number of balloons held by the clowns altogether. The idea of the picture was to make it clear that the numbers that need to be multiplied are (20x10), (20x8), (3x10) and (3x8). The first two calculations correspond to the clowns sitting in the bus with the strings containing ten balloons and eight balloons and the second two calculations correspond to the clowns standing outside the bus holding strings with ten and eight balloons on them. The diagram helped children see that Eileen had neglected to calculate the (20x8) and the (3x10).

Discussion & Conclusion.

That multiplication was the operation needed to complete this task was uncontested by the children in the class. In developing their solutions to the problem, they drew on different categories of mathematical knowledge.

Some children relied on intuitive, context-specific knowledge and drew versions of the clowns in order to attempt solving the problem. Computational knowledge of multiplication was widely held, as would be expected by children who had completed third class. Although occasional errors were made, children had access to multiplication table cards if they wanted them so that even a lack of computational knowledge would not be a barrier to solving the problem.

Little evidence of the children using concrete knowledge emerged in the lessons. This may have been because no manipulable objects were made available to them to help them find an answer. Some children may have used the diagrams they drew as a form of concrete knowledge to support their solution but that is unclear from the data sources used.

Much evidence of principled knowledge emerged from the children. Donal and David showed understanding of place value, by separating the tens and the eights in eighteen (Donal) and in stating how numbers can be easily multiplied by ten (David). Several children had a tacit knowledge that numbers can be decomposed additively and of the distributive property of multiplication (e.g. Donal, David, Caitlin, Fintan and others). The teacher attempted to make this aspect more explicit by introducing the diagram with the clowns in and beside the bus. Students like Eileen have some understanding of the distributive property but it needs to become more explicit if she and others are to be ready to apply it in taking the next step to understand and use the long multiplication algorithm automatically. They need to grasp the principle that (a+b)*(c+d) requires multiplying a by both c and d and multiplying b by both c and d and not just multiplying a by c and b by d.

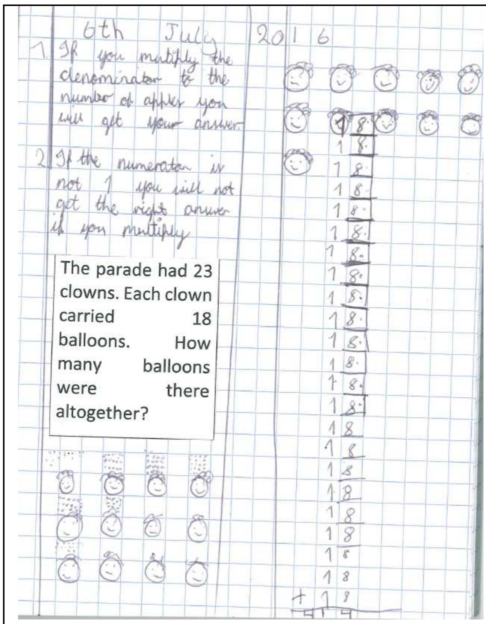

Figure 1. Christine’s work.

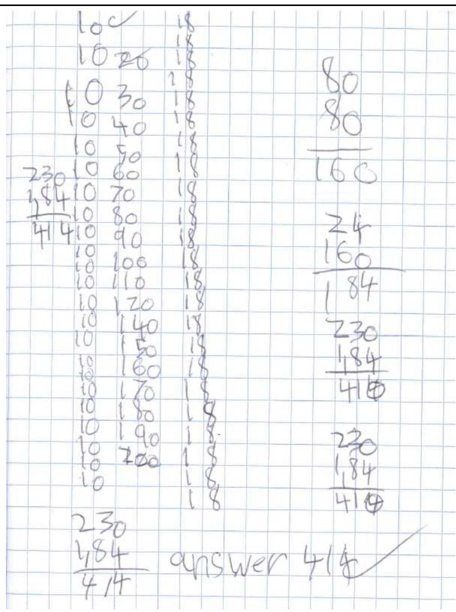

Figure 2. Donal’s work.

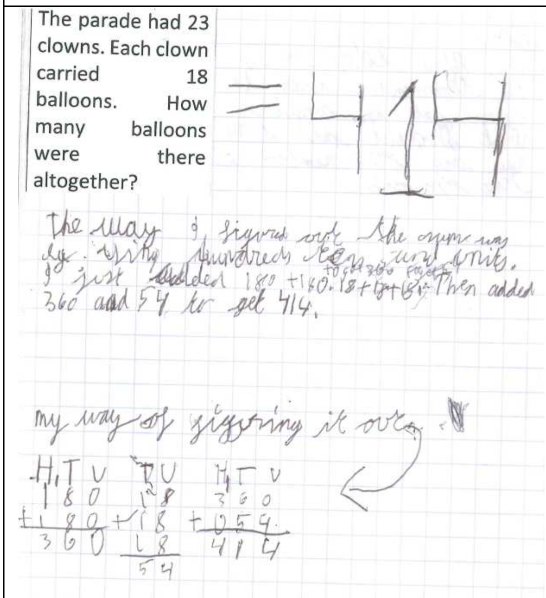

Figure 3. Fintan’s work.

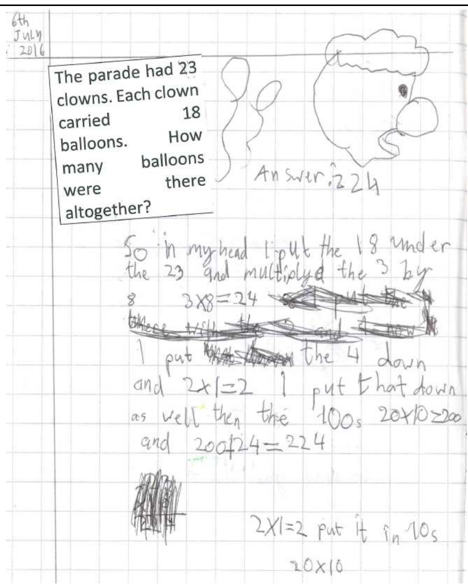

Figure 4. Eileen’s work.

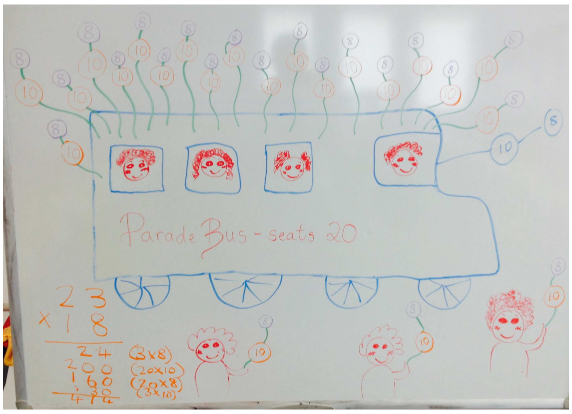

Figure 5. Visual representation of long multiplication problem.

At the end of the week around fourteen children were still either using repeated addition for similar multiplication tasks, using counting or not showing evidence of how they found their answer. Nine showed willingness either to apply the distributive property of multiplication or to at least attempt the conventional algorithm. Of those who were still applying the repeated addition algorithm, four had become more sophisticated in using it (moving closer to Donal's approach than to Christine's) following the class discussion documented here. This highlights the importance of children learning from each other through working on and explicitly discussing their completion of a task.

Two thirty-minute lessons is a very short time in which to introduce long multiplication to children. Nevertheless, the indications are positive that with repeated, continuous work over several days, many children would be well placed to learn to use the conventional algorithm to solve problems of multiplying a two-digit number by a two-digit number.

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