Tasks to develop flexible multiplicative reasoning

Joana Brocardo[1], Jean Marie Kraemer[2], Fátima Mendes[3], Catarina Delgado[4]

[1]Escola Superior de Educação, IPS de Setúbal and
UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; joana.brocardo@ese.ips.pt

[2]CITO, The Netherlands, jeanmariekraemer@gmail.com

[3]Escola Superior de Educação, IPS de Setúbal, Portugal; fatima.mendes@ese.ips.pt

[4]Escola Superior de Educação, IPS de Setúbal, Portugal; catarina.delgado@ese.ips.pt

The project 'Numerical thinking and flexible calculation: critical issues' aims to study students' conceptual knowledge associated with the understanding of the different levels of learning numbers and operations. We constructed an explorative instructional sequence that integrates the main types of contexts and activities that constitute the field of experience in which students develop what is called "multiplicative reasoning". We focus our analysis in the development of the first task of the sequence, illustrating how data analysis of students' solutions is used to reformulate the task.

Keywords: Task design, multiplicative thinking, flexible reasoning

Conceptual framework for multiplicative reasoning

We follow the idea that flexibility refers to the ability to manipulate numbers as mathematical objects, which can be decomposed and recomposed in multiple ways using different symbolisms for the same object (Gravemeijer, 2004; Gray & Tall, 1994). The key idea to develop a coherent and adaptive/flexible multiplicative reasoning consists in the integration of two core aspects/components of flexible mental calculation: (1) the personal arithmetical knowledge (Tall, 2013; Threlfall, 2002) that progressively emerges from a long term encapsulation process (Sfard, 1991) and (2) the idiosyncratic practice of flexibility (Tall, 2013), developed in the interrelated local situations of problem solving (Threlfall, 2002; 2009).

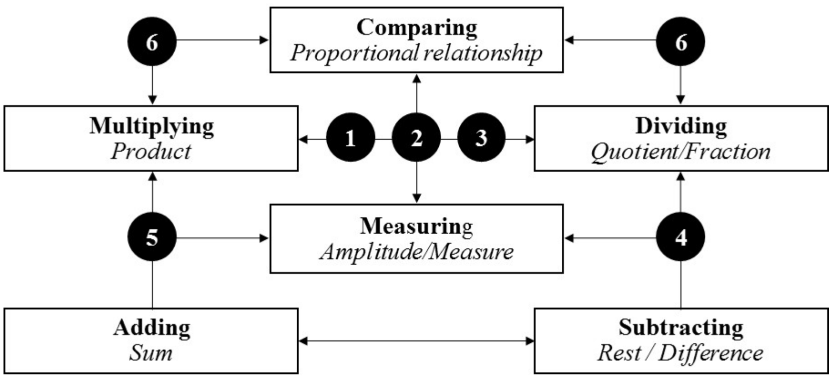

Figure 1: Conceptual field of the sequence

Figure 1 represents our first attempt to envision the result of this integrative process, used to develop a sequence of 9 tasks. We conceived one field of multiplicative reasoning as interrelated "conceptual schemes" that integrates procedural and conceptual knowledge that students have to gradually acquire along the sequence (Tall, 2013). Each of these schemes entails "imagining, connecting, inferring, and understanding situations in a particular way” (Thompson & Saldanha, 2003, p. 12).

In this figure, the number 1 represents the types of problems referred to in the literature (e.g. Greer, 1992) as a class of equal-groups situations modeled by multiplication and division.

Number 2 (Figure 1) represents dividing by repeated subtraction as the counterpart of multiplying by repeated addition. Phenomenologically viewed, dividing arises as what Freudenthal (1983) called “continually taking away”, which is conceived as the counterpart of repeated addition. The surprise of the remainder generates the big issue of dividing. Subtracting q times d from a number, means that what remains has to be smaller than d: $a = qd + r$ with $r < d$.

Number 3 (Figure 1) refers to the idea of division as inverting a multiplication. Children experience continually taking away as an annoying sequence of difficult calculations (for example: 61 – 5 = 56, 56 – 5 = 51; 51 – 5 = 46 …) that encourages them to adapt the conventional way of modelling by inverting the reasoning from the perspective of exhausting a to the perspective accumulating a.

Number 4 (Figure 1) represents the big idea of “segmentation” that emerges from structuring quantities (ratio and partitive division) and quantifying length, volume, time, etc. (measurement) in q units of d, with or without “remainder”. By envisioning the result of dividing and measuring, children learn to use the same principle and the same sequence of repeated addition or subtraction to represent, for example, to how many children you can give 4 oranges if you have 12 or to visualize a distance or length as 3 times 4 steps (Figure 2) on the same number line (Freudenthal, 1983; Thompson & Saldanha, 2003).

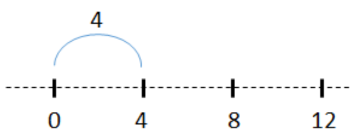

Figure 2: Segmenting by repeated addition

Number 5 (Figure 1) represents the idea of “accumulating”. The same representation on the number line shows the connection between multiplying and measurement: 3 times 4 oranges gets 12 oranges as 3 times 4 steps get 12 steps.

Finally, number 6 refers to the “multiplier”. Phenomenologically, comparing with the terms “double” and “half” precedes comparing multiplicatively as an arithmetical operation (e.g. Freudenthal, 1983). Children extend the use of the multiplier in statements such as “John has 3 times as many apples as Mary” before reasoning in terms of proportionality (scalar comparison).

Each designed task is integrated in a particular “point” of the web and should support gradual encapsulations and reification (Gray & Tall, 1994) and facilitate an interaction between *noticing* and knowledge, which is an essential aspect of the flexible mental calculation (Threlfall, 2002).

Key principles of designing task with focus on flexibility

Tasks or sequences of tasks are designed to embody mathematical knowledge and to improve students’ mathematical thinking (Ainley & Margolinas, 2013). The chosen approach prompts the

gradual building of the relational framework of number conceptions, operations and mathematical relationships involved in the field of multiplicative thinking, and the related acquisition of both the symbolic to express them and the ability to analyze, connects and manipulates them as mathematical objects. In this manner, we transcend the common focus on meaningful knowledge that stimulates adaptive expertise fostering that students come to act and reason in a "mathematical reality" (e.g. Freudenthal, 2003; Tall, 2013; Gravemeijer, Bruin-Muurling, Kraemer & van Stiphout, 2013), manipulating flexibly mathematical objects and relationships at hand through symbolic representation as professional mathematicians do (e.g. Bas, 2005).
We use the framework above to develop three kinds of tasks with a specific function and to articulate them transversally and vertically in cycles of mathematizations (Bell, 1993), using scheme-related relationships, their symbolic expression and representations of numbers that mirror multiplicative structures as potential junctions between acquired knowledge and skills. *Open task*s as "Prawn skewers" (Figure 3) prompt the exploration of a key idea (structure; relationship) that is relevant for a class of situations and the symbolic used to express it (e,g. Bell, 1993; Back, 2011). *Conventional* tasks focuses by turn on understanding how a particular relationship (theorem-in-action) of a scheme works and can be adapted in a limited class of situation. *Numerical* tasks (without context) encompasses the building of networks in which numbers as 60 form the junction between various concepts, operations and relationships (multiples of 2, 3, 4, 5, 6, 10, 12, 15) and the development of specific skills such as operating with the process-object symbolic and with mathematical objects such as product, multiple, operator. Key relationships tied to schemes of reasoning and classes of situations form the horizontal junctions between the tasks. These tasks are articulated vertically, taking into account the transitions from a lower (informal) to a higher (formal) level of reasoning, symbolizing and computing.
The task 'Prawn skewers' (Figure 3) exemplifies this approach. In this stage of instruction, Portuguese grade 3 students are building and memorizing the tables of multiplication and learning to use them in equal group situations. The task provides an opportunity to explore "partitioning" as a process of structuring quantities on the own level of understanding the relationships involved in equal group situations (junction [1], [2], [3], and [4] of fig. 1) and thinking about the numbers and number patterns in the available tables of multiplication.

We conjectured that the great majority of the children would notice that 61 is an odd number (ending by 1), near 60 (60+1 and/or 62-1). Focusing on 60, they would in first instance recognize it in a number that can be reached counting by ten (10, 20, 30, …) which suggests partitioning the pile of prawns by way of repeated addition (10+10+10+10+10+10). Students that operate on a higher level of understanding two-digit numbers would associate 60 with "six tens" seen as 6x10. After arriving at this point, one can explore other grouping possibilities, like varying additively or multiplicatively the number of prawns of each stick and, consequently, the number of sticks, connecting or not the new structure to the first one. Considering the current phase of development of grade 3 students, we expect difficulties with the symbolization of founded structures with products (e.g. difference between "6 times 10" written as 6 x 10 and "10 times 6" written as 10x6) and related misunderstanding in the communication about the transformation of one possible structure into another one (e.g. 6 times 10 into 12 times 5; 6x10=12x5). Finally, we expect that some students could first approach the task by describing the process of exhausting the pile of 61 prawns with an arithmetical sequence of repeated subtraction (junction [2]) and then (quickly) invert their modeling to avoid the (arising) computing difficulties.

Methodology

The project plan is based on a qualitative and interpretative methodology (Denzin & Lincoln, 2005) with a design research approach (Gravemeijer & Cobb, 2006). The preparation of teaching experiences is a crucial aspect of the project plan.

To prepare teaching experiences we design and reformulate mathematical tasks using a three step cyclic process: (1) design tasks, (2) analyze what children noticed in the numbers and how they use their knowledge about numbers and operations to solve the task presented along clinical interviews and (3) reformulate the previous task.

This text refers to one teaching experiment that involved 24 grade 2 students (age 7-8) and focuses on the students reasoning to solve the task 'Prawn Skewer' (Figure 3). The data was collected through video and audio recordings of the classroom work, researchers' notes, audio recordings of the preparation and reflection meetings with the school teacher involved.

The proposed task was designed and reformulated according to the data analysis of four clinical interviews, where students (4 students, 8 year old) solved a first version of the task (Figure 4) analyzed in Brocardo, Kraemer, Mendes and Delgado (2015).

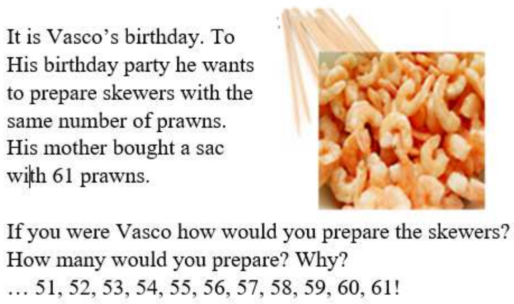

Figure 3: The tasks 'Prawn skewer' (reformulated)

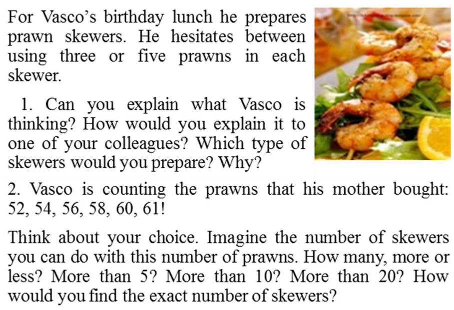

Figure 4: The tasks 'Prawn skewer' (first version)

Since the given alternatives in the first version (groups of 3 or groups of 5) seemed to hinder the *envisioning* of other ways of grouping, we decided to give students the freedom to experiment and evaluate different ways of grouping, taking into account two conditions: the given quantity of prawns and the freedom to invite more or less friends. We also `opened´ the illustration of the task, to stimulate the students' own constructions. Finally, by giving only the number of prawns, and by continuing to choose the `ugly´ number 61 (in de sense of Thompson & Saldanha, 2003), we created a problem that these students surely never encountered before.

Under these conditions, we expected that students would envision different possibilities of sticking a pile of 61 prawns, modeling from two ways of understanding "division". This is to say, reasoning in terms of exhausting the pile by a sequence of repetitive subtraction (ratio/measurement division), and/or reasoning from the converse idea of accumulating 61 by counting on n by n (division as converse of multiplication) (Freudenthal, 1982). In second instance, we expected that the choice condition of the task should stimulate students to compare envisioned ways of grouping, taking into account the relation between the multiplier (number of sticks) and the multiplicand (number of prawns in each stick). Exhausting, taking away a smaller set of prawns go together with making more sticks and subtracting a bigger set with obtaining less sticks. And, increasing the number of sticks by accumulating goes together with decreasing the number of prawns in each stick, as putting a bigger sets of prawns goes together with obtaining less sticks, while sticking less prawns provides more sticks. Finally, we expect a great variety of modeling, verbal explanation, and calculations, according to levels of memorizing the products/multiples of the tables, and understanding `multiplicity´ and `proportionality´ in the experienced contexts of multiplicative thinking (Figure 1).

Results

The analyses of the working sheets of the students and of some dialogues occurred in the classroom gave us a global idea of the patterns of reasoning that students used to solve this task.

Globally, we identified the following patterns of reasoning: trying with 18 sticks (they counted the sticks represented in the illustration); drawing the prawns' skewers one by one (with 18 sticks); putting 2 by 2 we will arrive to 60; putting 10 by 10 gives 60; adaptation of grouping by 10 making one skewer with 11 prawn (Figure 5); adaptation by "grouping by one"; putting 5 by 5 and/or 15 by 15 gives 60 (Figure 6); intuitive notion/feeling that putting by n would give 61.

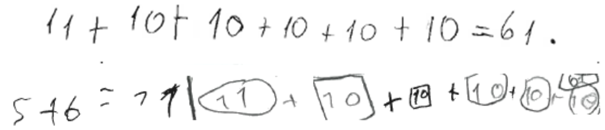

Figure 5: Adaption of grouping by tens, making one skewer with 11 prawns

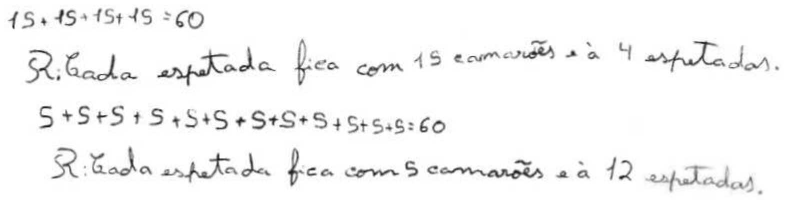

Figure 6: Putting by 15 and by 5 with verbal description of the result

The most frequent way of modeling was trying to reach 60 arithmetically, by repeated addition. Some students shortcut their long addition by mean of doubling consecutively the terms (Figure 7).

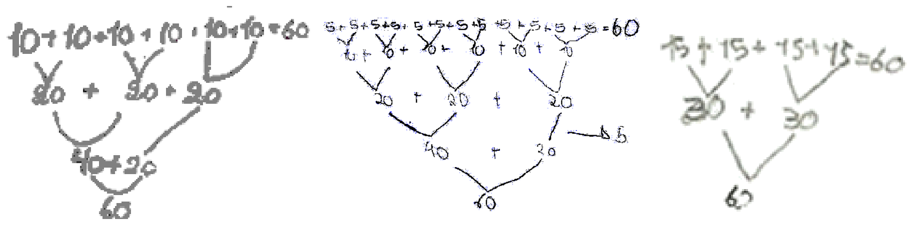

Figure 7: Shortcutting by successively doubling as way of controlling / justifying

Some dialogues suggest that some pairs of students are jumping to 60 in a kind of mental sequence of multiples, without keeping track. Having arrived at 60, they must then count the numbers of "tens" to derive the multiplier from their long addition.

Some dialogues illustrate how the confusion between *multiplier* and *multiplicand* troubles the communication around the way of reasoning and the corresponding way of counting/calculating.

Some pairs of students initially understood that they could choose the number of guests but failed to connect the given 61 prawns with the 18 sticks they counted in the picture. In a short dialog with the teacher, they envisioned ways of grouping with units, leaded by a kind of intuition (feeling) that the repeated addition they had in mind would give 61. Vera was thinking about 11 + 11 + 11 …, and moved to 5 + 5 + 5 … as response on the teacher´s question "Why 11?". Her colleague Martin was expecting that putting prawns by 7 would give 61, "since 7 + 7 = 14 and 14 + 7 = 21".

José drew the 18 sticks and then ´sticks' 4 prawns in each (Figure 8). He then symbolized the represented accumulation with the long addition 4+4+4 … and tried to determine the total number of prawns, by doubling 4 and 8, transforming 4+4+4+4 … (18 times four) into 8+8+8... (9 times 8) and 16+16+16+16+8.

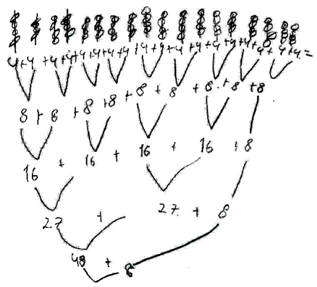

Figure 8: Wrong modeling by drawing 18 skewers with 4 prawns in each

Only one group used repeated subtraction (viewpoint of exhausting 61; Figure 9). They began to subtract 1 to obtain 60 but it seamed they changed their idea and begun to work with 61, subtracting 5 until they have a remainder smaller than 5.

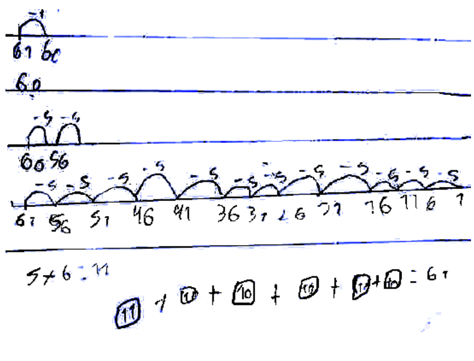

Figure 9: Modeling by repeated subtraction

Finally, two pairs of students associated the information of the story and the illustration of the task with the decomposition of 61 into two parts. One pair is thinking about a combination of 18 children (the number of sticks of the illustration) and 40 children (a multiple of 10). The other pair proposes "as 41 + 20 = 61, one has 41 and the other 20" and does not add any other justification.

Implications for task Design

From data analysis we may conjecture that the formulated instruction does not scaffold the expected mathematization including: (1) finding and representing adequately different forms of sticking the prawns, (2) choosing the preferred way, and (3) justifying this choice. It seems that “Why” is interpreted as an instruction to demonstrate and/or control that the modeling with the sequence of repeated addition indeed gives 60 prawns. This interpretation explains the spontaneous short-cutting of long additions (Figure 7). A solution to avoid this is to structure the task in an exploring phase asking explicitly to look for possible ways of sticking and an reflective one including the choice and the justification of the referred form of sticking.

On relating the choice of 61 as cardinal of the set, we can argument that the complexity of taking away, could explain the high frequency of the modeling by repeated addition and the single use of repeated subtraction. Since more students associate 61 with the near even number 60, and since the table of two and 4 are memorized, it would be meaningful to replace 61 by 62 to increase the chance that more students balanced between modeling by jumping back to exhaust 62 and jumping forwards to reach 62. We might expect that, being aware that this way of grouping would give a lot of `small´ sticks, more students would try to subtract a bigger quantity and finally move to jumping forwards to avoid annoying calculations.

Finally, since the representation of some stick may suggest some students to fix the multiplier and model the process directly by drawing all the set of prawns, stick by stick, we have to change the illustration to avoid the observed try-and-error approaches.

Implications for task design in the field of multiplication

In this text we explicit how we analyse the influence of contexts, numbers and pictures/images aiming to potentially facilitate *noticing* (Threlfall, 2002) relevant numerical relations and *envisioning* different approaches that emerge from noticing.

Another aspect of our analysis is related with the importance to propose non-conventional equal group problems to develop flexible multiplicative reasoning. In all conventional equal group problems, two values are given. Consequently, children learn to identify the ‘kind’ of problem from the story and the given numbers. Then, they solve it applying the standard scheme of reasoning. Children´s approaches and solutions of the task ‘Prawn Skewers’ show the advantage of the missing multiplayer and multiplicand as stressed by Back (2011). They have to adapt their common way of thinking to the unusual conditions of the task (Vergnaud, 2009) noticing numerical relations and envisioning different approaches. Some fixed the multiplier counting the sticks of the pictures. Others inferred that they had to fix the number of prawns on each stick and to look how many skewers can be made. The great majority envisioned the all process of sticking, using the knowledge that counting by ten leads to 60. A crucial advantage of these non-conventional problems is that the teacher can focus the reflection on the relationship between “continually putting prawns by *n*” and “taking again and again *n* prawns” and the use of asymmetric role of the multiplier and the multiplicand to symbolize both process.

References

Ainley, J., & Margolinas, C. (2013). Accounting for student perspectives in task design. In C. Margolinas (Ed.). *Task Design in Mathematics Education*. Proceedings of ICMI Study 22 (pp. 151-152). Oxford.

Back, J. (2011). Difficulties with division. *Primary Mathematics. February 2011.*

Bell, A. (1993). Principles for the design of teaching. *Educational Studies in Mathematics*, *24*(1), 5-34.

Brocardo, J. Kraemer, J. M., Mendes, F., & Delgado, C. (2015). Developing flexible-adaptive reasoning and computing: Pedro's understanding of the task "Prawn skewers". *Journal of the European Teacher Education Network 2015*, Vol. 10, 86-100.

Denzin, N., & Lincoln, Y. (Eds), (2005). *The Sage handbook of qualitative research*. Thousand Oaks, CA: Sage.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6* (2), 105-128.

Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. Akker, K. Gravemeijer, S. McKenney & N. Nieveen (Eds.), *Educational design research* (pp. 45-85). London: Taylor Francis Group.

Gravemeijer, K., Bruin-Muurling, G, Kraemer, J.M. & van Stiphout, (2013). Shortcomings of Mathematics Education Reform in the Netherlands, A Paradigma case? *Mathematical Thinking and Learning, 18*(1)*,* 25-44.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116- 140.

Greer, B. (1992). Division as Models of Situations. In Douglas A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning*, (pp. 276 – 295). New York: Macmillan Publishing Co.

Sfard, A. (1991). On the dual nature of mathematics conception: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Tall, D. (2013). *How humans learn to think mathematically: exploring the three worlds of mathematics*. Cambridge University Press.

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50*(1), 29-47.

Threlfall, J. (2009). Strategies and flexibility in mental calculation. *ZDM Mathematics Education, 41*(5), 541-555.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*, 83-94.