“Pépite” is an online automated assessment concerning elementary algebra for students in secondary education (12-16 year-old) in France. Pépite was initially developed for students at the end of compulsory school in France (16 year-old). At CERME9, we presented its transfer at different school levels and illustrated it with the design of Pépite test for grade 8th students. Information provided by Pépite allows identifying students’ consistent reasoning and calculation in order to organize teaching corresponding to students’ learning needs. In this paper, we focus on the use of Pépite test for grade 8th students to learn the domain of equations. We defined an epistemological reference of the algebraic domain that allows us not only to build the tasks selected for the test and to analyze students’ responses but also to propose suitable courses adapted to students’ learning needs.

Keywords: diagnostic assessment, Information and Communication Technology (ICT), elementary algebra and the domain of equations, student’s profile, teaching suggestions.

Context of the study

This paper deals with the issue “Digital assessment of and for learning” of TWG16 “Students Learning Mathematics with Technology and Other Resources”. Since the 1990s, our team has developed several multidisciplinary projects (Delozanne & al., 2010; Grugeon-Allys & al., 2012) concerning the design, development and use of online tools for diagnostic assessment and student learning. One of these tools, named “Pépite”, is relevant for learning elementary algebra for students of secondary education (12-16 years) in France. We have disseminated Pépite online tool on platforms largely used by teachers and students.

In this paper, we deal with the use of Pépite online assessment for learning the domain of equations for grade 8th students. First, we remind the theoretical foundations of Pépite online assessment. Then, we illustrate it with Pépite assessment for grade 8th students in France (13-14 years). We specify both the didactical model and the computer model that automatically generates generic

1 Pépite tools are available on LaboMep platform (developed by Sésamath, a French maths’ teachers association): [http://www.labomep.net/](http://www.labomep.net/) and on WIMS environment (an educational online learning platform spanning learning from primary school to the university in many disciplines).
tasks, analyses students’ work and provides descriptions of students’ profiles. Finally, we discuss
the potentialities of Pépite online assessment to propose suitable courses adapted to students’
learning needs for the domain of equations for grade 8th.

The theoretical and methodological framework

In the educational system, assessment is a complex issue. Usually, assessment results are generated
from standardized and psychometric models. Studies highlight the strengths and limitations of such
approaches for making instructional decisions (Kettelin-Geller & Yovanoff, 2009). To identify the
features of appropriate online assessment for learning, we have chosen both a cognitive and
epistemological approach and also an anthropological approach, which potentialities are described
in Grugeot-Allys & al. (2012).

Epistemological and cognitive approach

Designing a test requires selecting a set of tasks that enables the assessment to be realized. We
agree with Vergnaud who stated, “Studying learning of an isolated concept, or an isolated
technique, has no sense” (Vergnaud, 1986, p. 28). Furthermore, Vergnaud introduced a strong
assumption: dialectics between genesis of a student's knowledge and mathematical knowledge
structure. Beyond a quantitative analysis of responses, we have to define a qualitative didactical
analysis (based on a collection of students’ responses to the tasks) to identify the type of procedures
and knowledge used by students in solving the tasks. To provide descriptions of a student’s
consistent reasoning, it is necessary to define a reference for modelling the mathematical
competence, in a mathematical domain, at a particular school grade.

Anthropological approach

The epistemological approach is not sufficient so as to take into account the impact of the
institutional context on students’ learning. According to the anthropological approach, mathematical
knowledge is strongly connected to the institutions where it has to live, to be learnt and to be taught;
it is strongly connected to mathematical practices (curricula, etc.). Chevallard (1999) analyses
knowledge in terms of praxeology, that is to say in terms of type of tasks, techniques used to solve
these tasks, technological discourses developed in order to produce, explain and justify techniques,
and last, theory that justifies technological discourses.

A reference epistemological praxeology for algebraic knowledge

For a given mathematical domain, we defined a reference epistemological praxeology (Garcia,
Gascon, Higueras & Bosch, 2006) that makes it possible to a priori design the features of an
appropriate assessment. For algebraic knowledge, such reference is based on results from didactics
of algebra (Chevallard, 1989; Artigue & al., 2001; Kieran, 2007). In its tool dimension (Douady,
1985), there are tasks for generalizing, modelling, putting into equation, proving. In its object
dimension, there are tasks focussed on calculus with algebraic expressions (calculating, substituting
a number for a letter, developing, factorizing) or equations (solving). This reference makes it
possible to define appropriate technology for an intelligent and controlled algebraic calculus, based
on equivalence of algebraic expressions and the dialectic between numeric and algebraic treatment
modes.
The domain of equations for grade 8th students

The three following types of tasks are specifically related to equations (we will give some precise examples on the paragraph below about experimentations):

- Modelling and putting a problem into equation (tool dimension). These tasks motivate the production of an equation in order to solve modelling problems and require semiotic conversions (Duval, 1993).

- Solving an equation thanks to an algebraic technique; proving that two equations are equivalent (object dimension). These tasks use the concept of equivalence and require transformational activity (Kieran, 2007).

- Testing if a number is a solution of an equation; identifying the degree of an equation (object dimension). These tasks are based on substitution and polynomial properties.

Features of Pépite online assessment

The Pépite online diagnostic assessment is based on a reference epistemological praxeology of the algebraic domain, both for designing tasks and for analyzing responses to the test. We will rely on the Pépite test for grade 8th students.

The didactical model

Pépite test

The test (targeting 13-14 years old students) is composed of 10 diagnostic tasks and 22 individual items covering the types of tasks defined below (Table 1). The tasks may be multiple-choice or open-ended items (Figure 1).

<table>
<thead>
<tr>
<th>Types of tasks</th>
<th>Number of items</th>
<th>Test items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculus</td>
<td>4 / 22</td>
<td>7.1 / 7.2 / 8.1 / 8.2</td>
</tr>
<tr>
<td>Producing numerical expressions</td>
<td>1 / 22</td>
<td>5</td>
</tr>
<tr>
<td>Producing algebraic expressions</td>
<td>2 / 22</td>
<td>3.1 / 6</td>
</tr>
<tr>
<td>Translation or recognition</td>
<td>14 / 22</td>
<td>1.1 / 1.2 / 1.3 / 2.1 / 2.2 / 2.3 / 3.2 / 4.1 / 4.2 / 9.1 / 9.2 / 9.3 / 9.4 / 10</td>
</tr>
<tr>
<td>Problem solving in different mathematics frameworks</td>
<td>1 / 22</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 1: Organization of the 8th grade level test in terms of types of tasks

Exercise 6: Proof and calculation program

A student says to another student: “You will always find the same result if you take a number, you add 6 to that number, you multiply the result by 3, you subtract triple the initial number”. This affirmation is it true for any number? Justify your answer.

Justification

Result

The affirmation is true for any given number: true or false?

Figure 1: Example of generalization task
Responses analysis: the multidimensional model of algebraic assessment

At the local assessment level (for each task), student’s responses are not only evaluated as correct/incorrect, but also according to their technological discourse, that justifies the techniques. The analysis concerns validity of response (V) and seven dimensions: meaning of the equal sign (E), algebraic writings produced during symbolic transformations (EA), numerical writings produced during symbolic transformations (EN), use of letters as variables (L), algebraic rationality (J), connections between a semiotic register to another (T) and skills with negative and decimal numbers (N) (Table 2) (Grugeon-Allys, 2015). We code the responses with assessment criteria, which depend on knowledge and reasoning involved in the techniques.

We illustrate the multidimensional model of algebraic assessment on the task “Proof and calculation program” (Figure 1). In order to solve this task, two a priori strategies are possible: an arithmetic strategy using a number or an algebraic strategy mobilizing a variable. Several incorrect techniques can illustrate an arithmetic strategy (Table 3) according to the rules used to translate or transform numeric expressions. Algebraic strategy may be incorrect (J3) if the conversion rules (T3 or T4) or algebraic transformation rules (EA3 or EA4) are inadequate (Table 4).

---

Table 2: The multidimensional model of algebraic assessment (partial view)

<table>
<thead>
<tr>
<th>Assessment dimensions</th>
<th>Assessment criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>Validity of response</td>
<td>V0: No answer</td>
</tr>
<tr>
<td></td>
<td>V1: Valid and optimal answer</td>
</tr>
<tr>
<td></td>
<td>V2: Valid but non optimal answer</td>
</tr>
<tr>
<td></td>
<td>V3: Invalid answer</td>
</tr>
<tr>
<td></td>
<td>Vx: Unidentified answer</td>
</tr>
<tr>
<td>Algebraic writings produced during symbolic transformations</td>
<td>EA41: Incorrect rules make linear expressions a^2-&gt;2a</td>
</tr>
<tr>
<td></td>
<td>EA42: Incorrect rules gather terms</td>
</tr>
<tr>
<td>Connections between a semiotic register to another</td>
<td>T1: Correct translation</td>
</tr>
<tr>
<td></td>
<td>T3: Incorrect translation taking into account the relationships</td>
</tr>
<tr>
<td></td>
<td>T4: Incorrect translation without taking into account the relationships</td>
</tr>
<tr>
<td></td>
<td>Tx: No interpretation</td>
</tr>
</tbody>
</table>

Table 3: The multidimensional model of algebraic assessment (partial view)

<table>
<thead>
<tr>
<th>Solutions</th>
<th>Reasoning and technological discourse</th>
<th>Coding</th>
</tr>
</thead>
</table>
| For number 5  
(5 + 6) × 3 – 3 × 5 =18 | Correct arithmetic strategy with global expression that uses parenthesis | V3, L5, EA1, J2, T1 |
| For number 5  
5 + 6 = 11; 11 × 3 = 33; 3 × 5 = 15; 33 – 15 = 18 | Correct arithmetic strategy with partial expressions | V3, L5, EA1, J2, T2 |
| For number 5  
5 + 6 × 3 - 3 × 5 = 8 | Erroneous arithmetic strategy with global expression that uses no parenthesis | V3, L5, EA3, J2, T3 |

---

2 Contrary to usual practices in assessment, we do not attribute a code by technique for each task. This would lead to a multiplicity of codes on various tasks and would be unusable for a cross analysis on all the tasks of the test.
Table 3: *A priori* analysis for arithmetic strategies

<table>
<thead>
<tr>
<th>Solutions</th>
<th>Reasoning and technological discourse</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 6) \times 3 - 3 \times x) [\times] (3x + 18 - 3x) [\times] (= 18)</td>
<td>Correct algebraic strategy with global expression that uses parenthesis</td>
<td>V1, L1, EA1, J1, T1</td>
</tr>
<tr>
<td>((x+6) \times 3 = 3x + 18;) [\times] ((3x + 18) - 3x = 18;)</td>
<td>Correct algebraic strategy with calculus by step (procedural aspect)</td>
<td>V2, L1, EA1, J1, T2</td>
</tr>
<tr>
<td>(x + 6 \times 3 - 3x) [\times] (= x + 18 \times 3x) [\times] (= 2x + 18)</td>
<td>Erroneous algebraic strategy with global expression that uses no parenthesis</td>
<td>V3, L3, EA32, J3, T3</td>
</tr>
<tr>
<td>((x + 6) \times 3 = 3x + 18;) [\times] (21x - 3x = 18x;)</td>
<td>Erroneous algebraic strategy with calculus by step (procedural aspect)</td>
<td>V3, L3, EA42, J3, T4</td>
</tr>
</tbody>
</table>

Table 4: *A priori* analysis for algebraic strategies

Student’s profile, groups and differentiated strategies

The *Pépite* diagnostic assessment proposes both individual and collective assessment. The individual assessment, at the global assessment level (on a set of tasks), build the student’s profile which aims to identify features of algebraic knowledge and skills for the seven dimensions. The collective assessment locates a student on a scale with four components: skill in Algebraic Calculus (CA), skill in Numerical Calculations (coded CN), Use of Algebra for solving tasks (UA) and flexibility in Translating a semiotic register to another (TA). For each of those four components, different technological levels and appropriate benchmarks have been identified (Grugeon-Allys & al., 2012). Regarding to a class, students are divided into three groups according to their skill in Algebraic Calculus: CA1 (group A) - reasoned and controlled calculation preserving the equivalence of expressions -, CA2 (group B) - calculation based on syntactic rules often in blind, not always preserving the equivalence of expressions - and CA3 (group C) - meaningless and non-operative calculation. Therefore, for a given learning objective, it is possible to assign tasks to each group, depending to didactical variables related to the associated technological levels (Delozanne & al., 2010, Pilet & al., 2013).

The computer model

An iterative process between educational researchers, computer scientists and teachers was used to design and test different *Pépite* prototypes in order to improve the didactical model. We defined the conceptual IT model of classes of tasks, which allows characterizing equivalent tasks (Delozanne & al, 2008). The software *PépiGen* (Delozanne & al., 2008) automatically generates the tasks and their analyses, at different grade levels. It uses *Pépinière*, a computer algebra system, to generate anticipated student correct or incorrect answers (according to the *a priori* analysis). *Pépite* automatically calculates a student’s profile as well as profiles for groups of students. According to a learning objective defined by the teacher, *Pépite* generates tasks adapted to the related technological levels (Grugueon-Allys & al., 2012).
Results and discussion

The information provided by Pépite diagnostic assessment allows the teacher to identify students who have close profiles in algebra. Then, Pépite automatically generates differentiated routes corresponding to these algebraic profiles. As mentioned above, these routes were designed on the basis of a reference epistemological praxeology.

Differentiated routes for learning equations

Three differentiated routes were created concerning equations. The first one “Motivating the production of an equation and solving it with an equation solver” motivates the production of an equation. It includes tasks like “equalizing two calculation programs” (see the example below). Students have to solve them thanks to an equation solver. The second one “Algebraic resolution of an equation” requires technologies for solving equations by algebraic methods (by using the concept of equivalence of equations). In the last route “Algebraic resolution of problems that lead to an equation”, tasks about putting a problem into an equation and solving it – like “equalizing two perimeters of dynamic figures” – are proposed.

We give now two examples of tasks for the first route. The first one aims to introduce equations and to highlight the inadequacy of arithmetic techniques to solve problems of first degree. As we can see, thanks to a thoughtful choice of the didactic variables, this task prevents arithmetic strategies – because of the presence of the unknown in both calculation programs – or “trial and errors” methods – because the solution of this problem which is $\frac{7}{3}$ cannot be easily obtained by successive trials. Algebraic techniques are necessary.

<table>
<thead>
<tr>
<th></th>
<th>For groups A, B and C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Program A</td>
<td>Program B</td>
</tr>
<tr>
<td>Choose a start number</td>
<td>Choose a start number</td>
</tr>
<tr>
<td>Multiply it by 3</td>
<td>Multiply it by 6</td>
</tr>
<tr>
<td>Add 5 to the result</td>
<td>Subtract 2 to the result</td>
</tr>
</tbody>
</table>

Table 5: Task for motivating the production of an equation and solving it with an equation solver

The second task is differentiated (Table 6) to take into account students' algebraic activity and makes the students work on semiotic conversions (from the representation register of algebraic writing to the representation register of calculation program). Differentiation relies on didactical variables: the left member of the equation for group A is a product and solving the equation needs to use the distributive property, while the equation for groups B and C do not require it to be solved. Moreover, multiplication signs are let implicit in the equation for group A; they are not for groups B and C.

<table>
<thead>
<tr>
<th></th>
<th>For group A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For groups B and C</td>
</tr>
<tr>
<td>Write a problem with two calculation programs that corresponds to the equation $2(x + 7) = 5 - 3x$.</td>
<td>Write a problem with two calculation programs that corresponds to the equation $2 \times x + 7 = 5 - 3 \times x$.</td>
</tr>
</tbody>
</table>

Table 6: Task for working on semiotic conversions solver
Experimentation in a grade 8th class

We now present the results of an experimentation carried out in 2016 with a mathematics teacher we will call M2. M2 has been working in a REP establishment (high-priority education network) since three years. We chose him because he is not an expert. After an observation phase (6 hours) of his teaching practices, we proposed him a whole sequence about equations that takes into account the main epistemological aspects of the reference epistemological praxeology. M2 was free to adapt this sequence to his practices; however, both teacher and researcher have worked together to think the implementation in the classroom.

M2 should introduce equations in his grade 8th class. First, his 20 students (14 years old) passed the Pépite test. They then have been divided into three groups A, B and C. Only one student was in-group A – his algebraic activity was considered as appropriated. The others students belonged to groups B (15 students who can calculate correctly expressions but without using semantic rules) and C (4 students who do not understand the calculus on algebraic expressions). M2 proposed to his students the three routes mentioned above, in the same order. Due to the fact that most of his students were in group B (15/20), M2 chose to give the same tasks to the whole class. After working on the three routes, the 20 students passed a writing test on equations. We chose to focus on two tasks of this test to present our results. The first task was about solving three first-degree algebraic equations. Depending on the equation they solved, 7 to 11 students among the 20 students found the correct solutions. We particularly studied how many students used an algebraic technique. We observed that 17 out of 20 students solved the equations using the equivalence of equations. Even if they did not find the right solution, they had a strategy and transformed the equations in order to “eliminate” the unknown; they respected the concept of equivalence to do so. For the second task, equalizing two calculation programs (as presented above in table 5), 11 out of 20 students succeed for putting the problem into an equation.

Discussion

The Pépite assessment tool, based on an epistemological reference of the algebraic domain, allows the teacher to identify students’ consistent reasoning and calculation in order to plan differentiated courses adapted to grade 8th students’ learning needs for the domain of equations. The mathematics routes tested in our experimentation seemed to have effects on the students’ technological level: most of them used algebraic techniques to put a problem into equation. But this experimentation only concerns one teacher. So, in the ERASMUS + project “Advise me” which has just started in September 2016, we will carry out a larger scale experimentation.

We intend to validate these results for the field of arithmetic of integers for grade 3-4 pupils. Grapin (2015) carried out a multidimensional model of assessment for this new domain in elementary school. She defined an epistemological reference of arithmetic of integers to design an assessment tool in order to define pupils’ profiles and to highlight the epistemological aspects of arithmetic to work according to pupils’ learning needs. She organized an experimentation to study the evolution of pupils’ profiles according to differentiated routes adapted to students’ learning needs for the domain of arithmetic of integers. Data analysis is underway.
References


