How are Calculus notions used in engineering? An example with integrals and bending moments

Alejandro S. González-Martín[1] and Gisela Hernandes Gomes[2]

[1]Université de Montréal, Canada; a.gonzalez-martin@umontreal.ca;

[2]Université de Montréal, Canada; gisela.hernandes@umontreal.ca

Research has shown that mathematics courses in engineering programmes present students with a number of difficulties, some of which stem from a disconnection between mathematics course content and the professional activity of Engineers. Using tools from the anthropological theory of the didactic (ATD), we examine how the drawing of bending-moment diagrams is introduced in a classic textbook used in engineering programs. Although the notion of integral is used to teach this topic, the techniques used rely mostly on geometrical considerations (and not on integral techniques or theorems), and the justifications provided are a mix of (incomplete) mathematical discourse and professional justifications, with implications for students' learning.

Keywords: Mathematics for engineers, ATD, praxeology, integral.

Introduction and background

Mathematics is an important subject in many scientific and technological fields, including engineering. However, the difficulties university students face in their mathematics courses can lead them to abandon their professional aspirations (Ellis, Kelton, & Rasmussen, 2014). Research in engineering and mathematics education has shown that these difficulties manifest themselves in at least two points in a student's learning pathway. First, researchers have stated that students find the progression from secondary to tertiary education to be very difficult, especially when it comes to mathematics (Rooch, Junker, Härterich, & Hackl, 2016), and that they possess unsatisfactory mathematical readiness for engineering courses (Bowen, Prior, Lloyd, Thomas, & Newman-Ford, 2007). Second, a disconnect between mathematics courses and professional courses in university engineering programmes has been identified. For instance, Loch and Lamborn (2016, p. 30) stated that "mathematics is often taught in a 'mathematical' way with a focus on mathematical concepts and understanding rather than applications. The applications are covered in later engineering studies." This disconnect may create a "gap in the students' ability to use mathematics in their engineering practices" (Christensen, 2008, p. 131). This gap can be aggravated by the fact traditional engineering courses are usually separated into two groups: basic science courses in the first two years (such as mathematics and physics), and technical courses specific to each area of engineering in later years. Regarding this, Winkelman (2009, p. 306) indicated that "the first 2 years are typically devoted to the basic sciences, which means that students may only encounter engineering faculty in the third year of study". Some effort has been made to bridge the gap between mathematical and engineering practices, for instance by linking basic mathematical methods to applications (Rooch et al., 2016) or by introducing courses on mathematical modelling and problem solving early on in engineering programmes (Wedelin, Adawi, Jahan, & Andersson, 2015). These initiatives seem to have positive effects on student learning.

Tertiary mathematics education research has identified a number of difficulties encountered by Calculus students; however, there is a lack of research on how teachers of professional engineering courses consider and use the mathematical tools taught in prerequisite mathematics courses. In general, it is expected that students in second- or third-year professional courses have grasped the mathematical notions taught in their earlier courses. We are interested in studying how Calculus notions – which students are expected to master – are used in professional engineering courses; in particular, whether they are used in the same way as in Calculus courses. Specifically, our research analyses the presentation of Calculus notions in a classic engineering textbook. We anticipate that this analysis will help Calculus teachers in engineering programmes understand how the notions they teach are used in higher-year professional courses, which may lead to a reflection on the connections (or lack thereof) between the content of Calculus courses and that of professional courses. In this sense, we adhere to Castela's (2016) position on the issue of choosing appropriate mathematics for professional-oriented programmes: "mathematicians need to take some distance with their own culture […]. They have to reconsider the following questions: which mathematical praxeologies are useful for such engineering or professional domains? What needs would be satisfied? Which discourse makes the mathematical technique intelligible? This is actually an epistemological investigation that we consider as a prerequisite to the design of mathematic syllabi for professional training programs" (p. 426).

Theoretical framework

Because we are interested in how mathematical notions are used in Calculus and professional engineering courses, we believe that an institutional approach is appropriate for our research. In particular, Chevallard's (1999) anthropological theory of the didactic (ATD) provides useful tools for analysing mathematical activity, since it considers that human activities are institutionally situated, and, consequently, so is knowledge about these activities (Castela, 2016, p. 420).

A key element is the notion of *praxeology* (or *praxeological organization*), which is formed by a quadruplet $[T / \tau / \theta / \Theta]$ consisting of a type of task to perform T, a technique τ which allows the completion of the task, a discourse (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. In analysing tasks, we identify the *practical block* (or *know-how*) which is composed of types of tasks and techniques. The *knowledge block* describes, explains and justifies what is done, and is composed of the technology and the theory. These two blocks are important elements of the anthropological model of mathematical activity that can be used to describe mathematical knowledge.

Our research identifies specific *praxeologies* present in professional courses; we analyse how Calculus notions are applied in these courses and whether this application reflects how the notions are usually presented in Calculus courses. In this case, analysing the *practical block* of these *praxeologies* allows us to identify specific tasks that require the use of Calculus notions, whereas analysing the *knowledge block* allows us to identify the justifications given in using these notions, and compare them with the justifications usually given in Calculus courses. We consider the work of Castela (2016), who identified that "when a fragment of social knowledge, produced within a given institution I, moves to another one I_U in order to be used, the ATD's epistemological hypothesis states that such boundary crossing most likely results in some transformations of knowledge, called transpositive effects" (p. 420). Her model (p. 424) proposes that in the boundary-crossing process, some (or all) elements of the original *praxeology* may evolve, and it ascribes the same level of importance to types of problems and techniques as to concepts and theories. However, unlike Castela, we do not analyse the same type of task in two institutions, but rather a single *praxeology* specific to engineering and the use of mathematical tools within it.

Methodology

As we stated in the introduction (agreeing with Castela, 2016), in order to analyse how mathematics are used to solve problems in a given professional field, we must first understand and define these problems. We believe this is best achieved in collaboration with professional practitioners. To determine how Calculus notions are applied in professional contexts in engineering courses, we contacted an engineering teacher who holds Bachelor and Master of Civil Engineering degrees. Over the past 28 years this teacher has taught a variety of professional engineering courses at Brazilian universities, in engineering programs that meet international standards. He has also enjoyed a career in structural systems and reinforced concrete since 1986, developing projects and serving as a consultant. We interviewed him in March 2016 to understand how he uses Calculus notions in his professional courses. The interview and post-interview exchanges covered his way of teaching, the books he uses and the course notes he produces, focusing on his way of presenting different notions. For the purposes of this paper, we have chosen to analyse the introduction of shear force and bending moment, and, specifically, how integrals are used to introduce this topic. At his university, shear force and bending moment are introduced in the second year of the programme, in the Strength of Materials for Civil Engineering course (students take Calculus in their first year). Three classic international textbooks are listed in the course syllabus (all translated into Portuguese), the main reference being the book by Beer, Johnston, DeWolf and Mazurek (2012).

The teacher indicated he primarily follows the structure of the main reference book in teaching shear force and bending moment. Therefore, this paper focuses on the book's content; we are currently analysing the complementary material provided to students, as well as the content of the interview, which will be the source of future papers. In analysing the textbook, we identified how notions are introduced, the type of tasks associated with them, and the type of *praxeology* developed, paying particular attention to the *practical* and *knowledge blocks* and the role of mathematical tools and discourse within these blocks.

It is also important to note that in the prerequisite Calculus course at this instructor's university, certain properties and results are proved while others are simply stated. For instance, the connection between the sign of the derivative and the monotonicity of the function (θ_1) is present and used in some tasks (such as the drawing of functions), as well as the connections between differentiability and continuity (θ_2).

Shear and bending forces: a summary

The content introduced in this part of the course is related to the analysis and design of beams, an important aspect of civil and mechanical engineering. Generally, loads are perpendicular to the axis of a beam (*transverse loading*), which produces bending and shear in the beam. These transverse loads can be concentrated (measured in newtons, pounds, or their multiples of kilonewtons and kips), distributed (measured in N/m, kN/m, lb/ft, or kips/ft), or both (Figure 1).

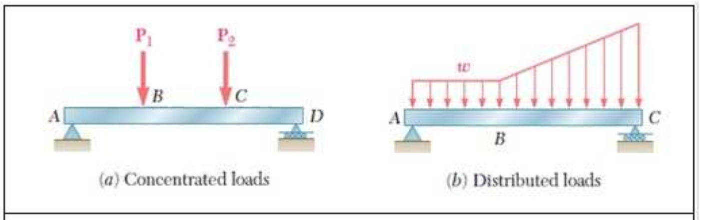

Figure 1: Types of loads (Beer et al., 2012, p. 346)

When a beam is subjected to transverse loads, any given section of the beam experiences two internal forces: a shear force (*V*) and a bending couple (*M*). The latter creates normal stresses in the cross section, whereas the shear force creates shearing stresses. Consequently, the criterion for strength in designing a beam is usually the maximum value of the normal stress in the beam.

Therefore, one of the most important factors to consider in designing a beam for a given loading condition is the location and magnitude of the largest bending moment. To determine this location, students are introduced to techniques for drawing bending-moment diagrams, defining *M* at various points along the beam and measuring the distance *x* from one end.

Data analysis and discussion

Although the main reference book develops its theoretical content in a well-structured way – which allowed us to grasp the notions presented – is it possible that students do not read it. Research examining how engineering students use their mathematics books seems to indicate that students pay little attention to theory, focusing instead on tasks (Randahl, 2012). We are not aware of research that looks at the way engineering students use their textbooks in professional courses.

The content addressing the drawing of bending-moment diagrams is presented in Chapter 5 (*Analysis and design of beams for bending*) of Beer et al. (2012). The chapter starts by introducing the different types of beam and loads, and the notions of load (*w*), *V*, and *M*. Section 5.1 introduces the relations between, and the directions of, the forces *V* and *M* in different sections of a beam, according to the type of load. In this section, calculations are made based on the idea that the sum of forces must equal zero, using formulae introduced earlier in the book. Sketches of bending-moment diagrams result in configurations such as the one shown in Figure 2. Obviously, someone with a background in Calculus could start to make a connection between the diagrams for *V* and *M*. However, this connection is not made in the textbook until section 5.2 (*Relationships between load, shear, and bending moment*).

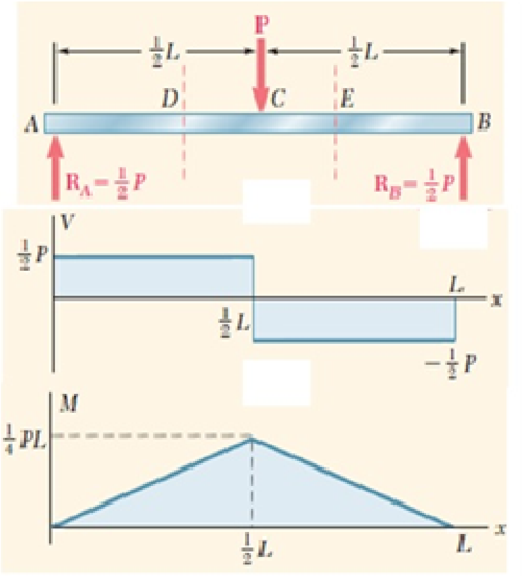

Figure 2: Example of bending-moment diagram for concentrated loads (Beer at al., 2012, p. 350)

The technique used in section 5.1 is quite rudimentary, but section 5.2 defines more explicitly (using derivatives and integrals – for this reason we focus on the content of this section) the relationships between w, V, and M to facilitate the drawing of bending-moment diagrams, which is the type of task (T_E) to solve. Section 5.2 presents a new *praxeology* (related to the one in section 5.1) that introduces the calculation of V and M at two adjacent points, x and Δx. Expanding on results from section 5.1, the authors arrive at $\Delta V = -w\ \Delta x$ and state: "Dividing both members of the equation by Δx and then letting Δx approach zero: $dV/dx = -w$. [This] indicates that, for a beam loaded as shown in [the given figure], the slope dV/dx of the shear curve is negative" (p. 360). We have two remarks about this. First, the book avoids the writing of limits. Including limits could help make a connection with mathematical *praxeologies* present in the prerequisite Calculus courses (for instance, when defining derivatives and shifting from Δx to dx). Even if the technology used to arrive at the final expression is based on content previously taught in a Calculus course, it is not certain that every student will make the connection, since tasks addressing the passage from Δx to dx are not very numerous in Calculus courses. Second, the book links dV/dx with the notion of slope, but (surprisingly) relates the latter to a single case (illustrated with a figure), rather than explaining it as a general principle using the technology θ_1 introduced in the Calculus course. This could lead some students to think that this connection between the slope of V and w applies only to the given figure. Although the notions (and their properties) introduced through T_E are defined using tools from Calculus, they are not explicitly linked to technologies (such as θ_1) derived from Calculus. Finally, the expression is integrated between points C and D to obtain: "$V_D - V_C = -\int_{x_C}^{x_D} w dx$" and "$V_D - V_C = -$ (area under load curve between C and D)."

In general, although the textbook uses elements of Calculus, it avoids explicitly using the kind of notation and properties that have been institutionalised in Calculus courses (such as θ_1 and θ_2 mentioned above). For instance, the books states: "[$dV/dx = -w$] is not valid at a point where a concentrated load is applied; the shear curve is discontinuous at such point" (p. 361). Here, the author avoids a clear statement about continuity and slope (available in θ_2). As Castela (2016) pointed out in a different context, we believe that the authors are seeking to develop another kind of knowledge, strongly correlated with a professional context. Employing techniques similar to those used to find V (and again, avoiding the writing of limits and saying instead "and then letting Δx approach zero"), the expression $dM/dx = V$ is deduced and the authors state: "[this] indicates that the slope dM/dx of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value (i.e., no concentrated load is applied). [It] also shows that $V = 0$ at points where M is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending". Interestingly, once again, the book's authors avoid using explicitly a technology derived explicitly from Calculus (θ_1), making it less likely that students will make the connection. They finally deduce that: "$M_D - M_C = \int_{x_C}^{x_D} V dx$" and "$M_D - M_C =$ area under shear curve between C and D."

We can see that the book avoids explicitly using properties previously institutionalized in Calculus courses, which leads to a kind of praxeology in which Calculus tools are written but geometric techniques are favoured. We do not mean to say these techniques are wrong; however, they could result in a knowledge gap, as some students may not recognise the same object (*integral*) that they encountered in their Calculus course. For instance, the first solved example (t_1) (Figure 3) presents a uniformly distributed load w. Using previous formulae, the reaction forces in the extremities are deduced (equal to $\frac{1}{2}wL$), which allows the deduction of $V_A = \frac{1}{2}wL$ and $V - V_A = -\int_0^x w dx = -wx$, leading to $V = V_A - wx = \frac{1}{2}wL - wx = w\left(\frac{1}{2}L - x\right)$. Note that the notation differs from that in the theoretical section, and the expression depends on the parameter w (introducing a technique τ_1 that differs from what was previously presented and that does not address the presence of w); however, the latter is not highlighted, and a graph is drawn (Figure 3c), taking for granted that students can interpret a graph depending on a parameter (ignoring students' known difficulties with parameters; e.g. Furinghetti & Paola, 1994). The maximum value of the bending moment is obtained by calculating the area under the positive triangular region $\left(M_{\max} = \frac{1}{2}\frac{L}{2}\frac{wL}{2} = \frac{wL^2}{8}\right)$, and the curve is hand-drawn (another technique that does not address that M has been introduced as the integral of V). The authors conclude with: "Note that the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree), and the bending-moment curve a cubic (third degree). The shear and bending-moment curves are always one and two degrees higher than the load curve, respectively. With this in mind, the shear and bending-moment diagrams can be drawn without actually determining the functions $V(x)$ and $M(x)$" (p. 362). A single case is used to introduce an important technological element (θ_E) that is helpful in solving T_E (drawn by hand), but this element is not justified in general, even though introducing V and M as integrals (showing that their coefficients can be deduced as primitives) would allow the use of a technology derived from the Calculus course for this justification. The book instead chooses to introduce a "rule" (θ_E) indicating that the student simply has to add one and two degrees, respectively, to draw $V(x)$ and $M(x)$. The next solved problem has students calculate (again using formulae from section 5.1) the values of forces in extremities of intervals as well as areas using geometry. Students are asked to draw by hand the bending-moment curve (Figure 4), even for cubic functions. This way, given the original diagram (Figure 4-top), students can deduce the value of V, which will be constant at certain intervals, and deduce its value at D and E specifically, while simply linking them with a straight line. Once a student has drawn the graph for V, it is possible to calculate the areas under each segment to deduce the values of M in B, C, and D, linking them by hand.

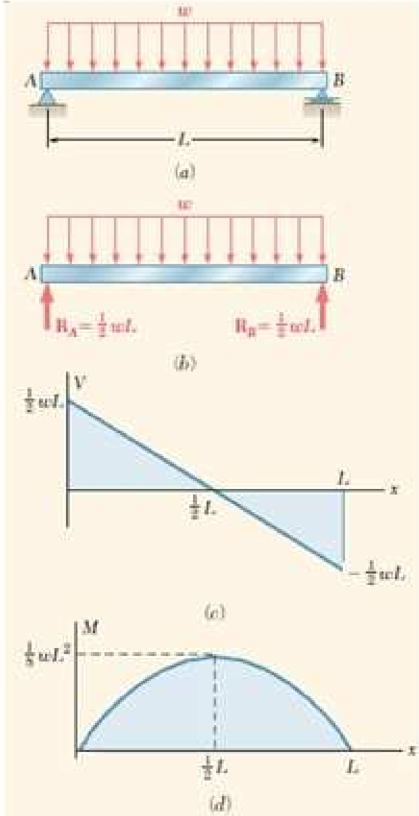

Figure 3: Solved example (Beer at al., 2012, p. 362)

In summary, the book introduces a *praxeology* to solve the problem of drawing bending-moment diagrams (T_E); however, although related notions are introduced using mathematical tools such as integrals, the technologies rely on implicit mathematical results without clearly identifying them, favouring a more professional perspective. The techniques presented are limited to calculating certain points on graphs and linking them using geometric properties, which hinders students' ability to make connections with the techniques and notions introduced in their Calculus course. Notions are presented as integrals but this fact is not explicit in the book's techniques nor in the technology; because it is possible to ignore the book's explanations when focusing on techniques, it is not certain that students will connect this content with content previously studied in Calculus courses. The book introduces a *praxeology* in which the *practical block* is clearly presented [T_E, τ_E], but where the *knowledge block* (mainly θ_E) mixes statements from mathematics and the engineering profession, leaving many facts implicit. Furthermore, this type of task does not justify all the content and techniques previously learned in Calculus courses regarding integrals.

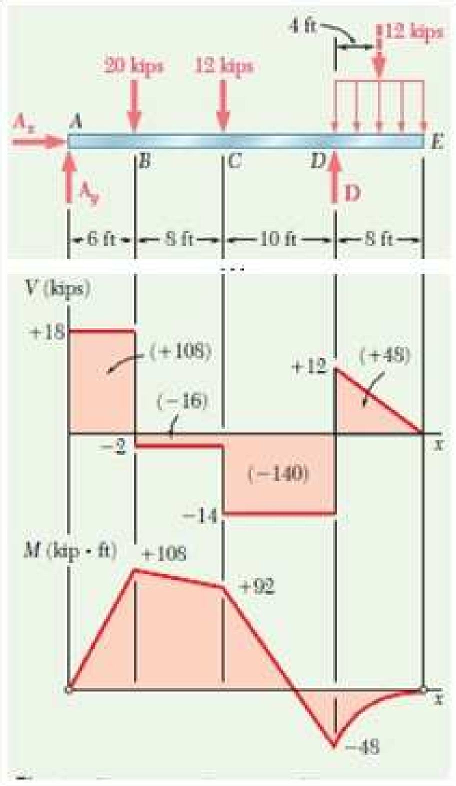

Figure 4: Solved example (Beer at al., 2012, p. 363)

Final remarks

In this paper we analysed the process of boundary crossing (Castela, 2016) of content related to integrals, and examined how this content is used as technique and technology in a *praxeology* proper to civil and mechanical engineering. The literature has identified disconnections between mathematics and professional engineering courses (Christensen, 2008; Loch & Lamborn, 2016) and our research has allowed us to pinpoint one of these disconnections. Furthermore, we believe the tools provided by ATD allow us to study *praxeologies* and identify the connectivities and disconnectivities between the content in mathematics courses and professional courses.

It may be argued that the study of integrals in engineering programmes is motivated by the simple fact that "engineers use integrals". However, we believe that the way integrals are taught in Calculus courses follows acknowledged *mathematics praxeologies* (those which are accepted and recognized by the institution of mathematics research; Castela, 2016, p. 421). These *mathematics praxeologies* ignore the use of integrals in professional courses. The crucial question, evoked in the introduction, of "what needs would be satisfied?" seems to be ignored by the *praxeologies* developed in Calculus courses, resulting in two different uses of *the same* object. We intend to analyse the entire content of the book related to sheer forces and bending moments, as well as the course notes, to provide a more detailed portrait of the use of integrals in this content. This work will be followed by further analysis of other engineering-related content, which will allow us to better understand the use of Calculus content by engineers and pinpoint possible gaps experienced by engineering students.

Acknowledgments

The authors wish to thank Januário Pellegrino Neto, the engineer who participated in this project, for sharing his knowledge and experience and for making himself available for the interview and our many follow-up questions. The research presented here was funded by grant 435-2016-0526 of the Social Sciences and Humanities Research Council (SSHRC) of Canada's *Insight* program.

References

Beer, F., Johnston, E. R., DeWolf, T., & Mazurek, D. F. (2012). *Mechanics of materials (7th edition)*. New York, USA: McGraw-Hill Education.

Bowen, E., Prior, J., Lloyd, S., Thomas, S., & Newman-Ford, L. (2007). Engineering more engineers – bridging the mathematics and careers advice gap. *Eng. Education*, *2*(1), 23–32.

Castela, C. (2016). When praxeologies move from an institution to another: an epistemological approach to boundary crossing. In R. Göller, R. Biehler, R. Hochmuth & H.-G. Rück (Eds.), *Proceedings of the KHDM Conference: Didactics of Mathematics in Higher Education as a Scientific Discipline* (pp. 153-161). Kassel, Germany: Universitätsbibliothek Kassel.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Christensen, O. R. (2008). Closing the gap between formalism and application — PBL and mathematical skills in engineering. *Teaching Mathematics and its Applications*, *27*(3), 131–139.

Ellis, J., Kelton, M. L., & Rasmussen, C. (2014). Student perceptions of pedagogy and associated persistence in calculus. *ZDM - The Int. Journal on Mathematics Education*, *46*(4), 661–673.

Furnghetti, F., & Paola, D. (1994). Parameters, unknowns and variables: a little difference? In J. P. da Ponte & J. V. Matos (Eds.), *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 368–375). Lisbon: University of Lisbon.

Loch, B., & Lamborn, J. (2016). How to make mathematics relevant to first-year engineering students: perceptions of students on student-produced resources. *International Journal of Mathematical Education in Science and Technology*, *47*(1), 29–44.

Randahl, M. (2012). First-year engineering students' use of their mathematics textbook - opportunities and constraints. *Mathematics Education Research Journal, 24*(3), 239–256.

Rooch, A., Junker, P., Härterich, J., & Hackl, K. (2016). Linking mathematics with engineering applications at an early stage – implementation, experimental set-up and evaluation of a pilot project. *European Journal of Engineering Education*, *41*(2), 172–191.

Wedelin, D., Adawi, T., Jahan, T., & Andersson, S. (2015). Investigating and developing engineering students' mathematical modelling and problem-solving skills. *European Journal of Engineering Education, 40*(5), 557–572.

Winkelman, P. (2009). Perceptions of mathematics in engineering. *European Journal of Engineering Education*, *34*(4), 305–316.