A case study on Finnish pupils' mathematical thinking: problem solving and view of mathematics

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In this article, the mathematical thinking of four Finnish pupils is reported using two temporally different data sets: problem-solving processes and view of mathematics. While the pupils seem similar on the surface level (high achievers, successful problem solvers, enjoy mathematics, motivated to learn mathematics), a closer look at their problem-solving processes and view of mathematics reveal very different strengths and weaknesses in their mathematical thinking. Most of the similarities in this study were found in individual pupils' problem-solving processes and view of mathematics.

Keywords: Problem solving, view of mathematics, affect, metacognition, meta-affect.

Introduction

Developing mathematical thinking is one of the key tasks for mathematics instruction in the Finnish curriculum (FNBE, 2014, 2004). And indeed, Finnish pupils have succeeded well in international studies that assess pupils' mathematical thinking (PISA and TIMSS; see e.g. OECD, 2014; Mullis, Martin, Foy, & Arora, 2012). However, the most recent national and international studies show that the mathematics performance of Finnish pupils is descending (e.g. Välijärvi, 2014; Rautopuro, 2013). Additionally to the alarming trend in mathematics performance, we know very little about Finnish comprehensive school pupils' mathematical thinking that go *beyond* paper tests. Thus, a qualitative research study was conducted with the aim of describing what characterises Finnish 15-year-old pupils' mathematical thinking.

On the way to describe what characterises Finnish pupils' mathematical thinking, the study reported in this article examines four high-achieving Finnish pupils' mathematical thinking through the intertwined relationships of problem-solving processes and view of mathematics. While some of the results of individual pupils' mathematical thinking have been discussed in previous publications (Viitala, 2013; 2015a; 2016a), the purpose of this paper is to bring the results together, and answer *what similarities and differences related to mathematical thinking can be found between these pupils*. With this question, we can reveal some of the possible trends in skills and competences that the Finnish high-achieving pupils might have in their mathematical thinking.

Theoretical framework

Developing pupils' mathematical thinking is in the heart of mathematics education, also according to the Finnish curriculum (FNBE, 2014). While research in mathematics education does not seem have a common understanding of the meaning of mathematical thinking, Schoenfeld (1992) recognised five aspects that are important in a study on mathematical thinking: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices. Similar findings have also been found in connection to literature on problem-solving performance (Lester

1994), and are also listed as part of final-assessment criteria in the Finnish curriculum (see FNBE 2014, pp. 433-434).

Similarly as the most recent theories on affect, mathematical thinking can be viewed through two temporally different aspects: state and trait (cf. Hannula, 2011; 2012). On one hand, mathematical thinking is always situational (state). Following Schoenfeld's (1992) categorisation, it is influenced by the pupils' knowledge base and heuristics, and guided by their metacognitive skills, affects and classroom practices. In this study, mathematical thinking is studied through problem-solving processes. In other words 'pupils' activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking' (Viitala, 2015a, p. 138).

Pupils' problem-solving behaviour is influenced by pupils' metacognition, affect and meta-affect that occur in a problem-solving situation. The successful application of problem-solving activities at the correct moment is a result of metacognitive skilfulness (e.g. van der Stel, Veenman, Deelen, & Haenen, 2010), affect influence the problem-solving situation for instance through the feeling of confidence, and meta-affect transforms individuals' emotional feelings (DeBellis & Goldin, 2006) and directs problem solving behaviour (Carlson & Bloom, 2005).

On the other hand, problem-solving situations can show patterns of thought that can be interpreted as signs of more stable ways of thinking. Some of these patterns can also be revealed through pupils' view of mathematics (see e.g. Viitala, 2016a). View of mathematics draws from psychological theories. It is a mixture of cognitive, motivational and emotional processes that include for instance beliefs, attitudes, values, feelings and motivation (Hannula, 2011; 2012). In this study, view of mathematics is studied through four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen, 1995, cf. Op't Eynde, de Corte, & Verschaffel, 2002).

Methods

Data collection

At the time of data collection, the four pupils (Alex, Daniel, Emma and Nora) were 15 years old and in their 9^{th} and final year of compulsory school in Finland. Additionally, they were all high achievers (mathematics grades between 9 and 10 on a whole number scale of 4 to 10).

The data was collected in three cycles over the course of three months. In each cycle, one mathematical task was solved in an ordinary classroom situation as a 'main task'. The pupils solved the tasks individually but they were allowed to talk about the tasks with a friend or ask for help from the teacher. In each cycle, the pupils were video recorded while they solved the task(s) in class and their solutions on paper were collected. Below, there is an example of a main task (School Excursion, OECD, 2006, p. 87).

> A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.
>
> Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.

Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

In each cycle, the pupils were interviewed individually. The interviews took place either on the same day, or on the next day after solving the task in the classroom. The interviews contained two parts. The first part concentrated on affective traits and treated the following themes: pupil's background, mathematical thinking, and pupil's view of mathematics (following the categorization of Pehkonen, 1995; see example questions in Table 1, Viitala, 2016a, p. 1295). This part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009).

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking	What does mathematical thinking mean? / How do you recognise it?
Mathematics	What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

Table 1. Interview themes and example questions.

The second part of the interview was about problem solving. The classroom data was used as stimuli when the pupil's problem-solving process was discussed. The pupils were asked to explain their thinking and actions during the problem-solving situation and additional questions were asked (e.g. what are you thinking now? Why are you doing so? What did you feel when you read the task? Did you think about your own thinking when solving the task?).

Finally, in each interview, the pupils were asked to assess their confidence before, during and after solving the problem, as well as their confidence in school mathematics using a 10 cm line segment (scale from 'I couldn't do it at all' to 'I could do it perfectly'). All interviews were video recorded.

Analysis

Following the state and trait aspects of the study, the analysis was divided into two sections: problem solving and view of mathematics. The problem-solving processes were analysed first by going through the problem-solving phases introduced by Carlson and Bloom (2005): orienting, planning, executing and checking (cf. Polya, 1957). Then the results on problem-solving behaviour were complemented with metacognitive activities (orientation, planning, evaluating and elaboration van der Stel et al., 2010), affect (state and trait, as well as cognition, emotion, motivation; Hannula, 2011; 2012) and meta-affect (DeBellis & Goldin, 2006) emerging in problem-solving processes. Finally, the pupils' confidence to solve the problems was analysed.

The first analysis of the pupils' view of mathematics followed the themes of data collection (Pehkonen, 1995). After condensing the results, a pupil profile was created to be used as background information about the pupil. Pupil profile is a short description of the pupil that is based on the pupil's mathematics grade, motivation to learn mathematics, and the core of his view of himself as a learner of mathematics (ability, success, difficulty of mathematics, and enjoyment of mathematics, following Rösken, Hannula, & Pehkonen, 2011).

In the end, the results of problem solving and view of mathematics were compared to see if there are similarities in pupil's problem-solving skills (state) and competences found through pupil's view of mathematics (trait). More details of the methods used in the study are reported for instance in Viitala (2015b).

Results

On a surface level, Alex, Daniel, Emma and Nora seem quite similar: they are all high achievers in mathematics, they enjoy mathematics, and they are motivated to learn mathematics (see excerpts in Table 2). They are also successful problem solvers, that is, they could solve all the problems given to them in the study and justify their answers and solutions. However, a deeper look at their problem solving and view of mathematics introduce four pupils with a very different skills and competences. In the following, the key results of each pupil will be introduced individually.

Alex is very fluent and thorough mathematics learner and problem solver. He can move naturally between different phases of problem solving. He is aware of his own thinking and fluent in explaining and justifying his cognitive and metacognitive actions in problem solving. Similarly, when explaining his learning of mathematics, he says he is actively seeking for connections between new knowledge and prior knowledge, and he is able to spontaneously give examples of this behaviour. He says he trusts his own thinking more than his calculations, and shows to be able to direct his behaviour according to his affects in problem solving. He is confident in school mathematics but in the interviews, he constantly compares his abilities to mathematics as a science and recognises that there is much more than school mathematics (more results in Viitala, 2013; 2016b).

Whereas Alex seems to be very fluent in every aspect of mathematical thinking studied in this research project, from a similar starting point, Daniel shows somewhat different strengths in mathematics. Unlike any of the three other pupils, he is extremely confident in mathematics. He says that mathematics is easy for him, and he shows to be very aware of his success in mathematics. His confidence seems to guide also his problem-solving processes. He is able to move fluently back and forth between problem-solving phases and is skilful in performing metacognitive acts. However, even though (or because of) learning mathematics and solving problems are easy for him, he cannot explain the processes he goes through in or for learning, and he has problems in explaining his problem-solving actions after the problem-solving situation. An illustrative example of this situation is Daniel's explanation about how he learns mathematics: pieces just click together or things become familiar (more results in Viitala, 2015b).

Similarly as in Daniel's case, also Emma's learning of mathematics and problem solving are strongly influenced by her confidence in mathematics, or more precisely, her lack of confidence. Because of the uncertainty in mathematics, for Emma, learning takes time and effort. She says she learns every topic as a separate entity, and she is able explain the steps that are needed for her to learn a new thing. Similarly, she uses a considerable amount of time for orienting and planning in problem solving. After understanding the problem and the given data, she is able to follow her plan through and check her solution. It seems that Emma‘s uncertainty in mathematics makes her work harder, and through hard work, she succeeds in mathematics. Moreover, she says that succeeding in mathematics and understanding it, makes it worthwhile studying. On the other hand, affect can also be an obstacle in her problem solving, since she does not seem to have efficient tools to overcome the feeling of getting stuck (more results in Viitala, 2015a; 2016a).

Also for Nora, learning mathematics takes time and effort but after learning something, applying is easy. She says that she is quite confident in mathematics and likes learning mathematics very much. She is capable in explaining her thinking and problem solving, and connecting mathematics to her own life. She also has a diverse view of mathematics as a science. In problem solving, she is flexible in directing her actions based on the affective states occurring in problem-solving situations. She is also fluent in moving between orienting, planning and executing in problem solving. However, given the choices she had made while planning, she is happy with the first answer she gets, and does not check her results (more results in Viitala, 2015a).

	Ability and success	**Difficulty of mathematics**	**Enjoyment of mathematics**	**Motivation to learn mathematics**
Alex	Confident in math; deserves the high grade: knows school math quite thoroughly	Learning 'a separate thing' is easy, connecting it to 'other things' might take time	Learning math is fun and interesting; routine learning is boring	Good grade and future studies, also understanding the issue at hand
Daniel	Very confident in math; can do math well; deserves the high grade (active learner, succeeds in tests)	Learning math is easy and it does not take much time or effort	Math is enjoyable, even fun	Math is needed through life; the most important school subject
Emma	Not confident in math; could not get a better grade in math	Learning math takes time and effort	Learning math is irritating and tiring; succeeding and understanding is fun	Wants to succeed in mathematics and be proud of herself; future studies
Nora	Quite confident in math; not perfect in math but deserves the high grade in school math (active learner, succeeds in tests)	Math can be easy or difficult, more on the easy side; learning takes time and effort, applying after that does not	Learning math is interesting, likes math very much	Good grade; wants to learn math

Table 2: Examples of pupils' own statements about their view of mathematics (cf. pupil profile).

Some reflections of the results

In addition to forming descriptions of pupils' mathematical thinking, and showing pupils' strengths, the study also revealed issues that pupils could work with in order to develop their mathematical thinking. For instance, even though Alex was fluent in problem solving and school mathematics, he did not relate the problems to real life and his view of mathematics outside school was quite limited (see Viitala, 2013, 2016b). Recognising mathematics more in his own life could enrich Alex's view of mathematics, and through that, also his understanding of school mathematics might develop.

Daniel, on the other hand, had problems explaining his thinking after the problem-solving situation and had similar problems with explaining his mathematics learning (see Viitala, 2015b). Problem solving and learning mathematics might be easy for Daniel in compulsory school, but what happens if (when) the situation changes? Becoming aware of his own learning and problem-solving processes could help him cope in new situations and develop his metacognitive skills not only in mathematics but also in other school subjects.

Emma's weak point was her uncertainty which she had turned into success in problem solving and learning of mathematics. She had overcome some of the uncertainty with the support of her family (see Viitala, 2016a). However, because she was not confident in mathematics, she learnt every topic in mathematics as its own entity, and did not connect it to prior knowledge. This might also hinder her learning. Hence, supporting Emma emotionally could open doors to more thorough learning and understanding of mathematics. Finally, Nora's results were not always correct, and both her activities and explanations showed that she does not evaluate her problem-solving process or check her results (see Viitala, 2015a). Supporting her to look back, and perhaps exposing her more to, for instance, open problems, might help her to become more reflective user and learner of mathematics.

Summary and discussion

The purpose of the paper was to answer the question what similarities and differences related to mathematical thinking can be found between the four Finnish high-achieving pupils. Mathematical thinking was studied through two temporally different data sets: problem-solving processes (state) and view of mathematics (trait). The results showed that the similarities between the pupils were found to be mainly on a surface level: all the pupils liked mathematics, were motivated to learn it, enjoyed doing mathematics and were successful problem solvers. However, after a deeper look into their problem-solving processes and view of mathematics, the study revealed a great deal of differences between the pupils, and showed different competences: Alex is a very conscious thinker and learner of mathematics, and excellent in justifying his thinking and actions in mathematics. Daniel is extremely confident and metacognitive skills are prominent in his problem solving. Emma is an unsure but very thorough problem solver and learner of mathematics. Nora is fluent in expressing her thoughts and connecting mathematics to real life.

In addition to the strengths found in these four pupils, the framework also revealed some of their weaknesses. The strengths, together with the weaknesses can be used to support individual pupils' development in mathematics. For instance, Alex seemed to see mathematics only as a tool to solve something and his view of mathematics outside school was quite limited (see Viitala, 2013, 2016b). This knowledge can be used to develop pupil's mathematical thinking. Four years after the data collection of this research project, I met Alex again. At this point, Alex was as a university student. He explained that only after realising the tool value that mathematics had for him, and learning that mathematics is not just calculations but also ways of thinking, he began to see mathematics everywhere in his real life, and he began to use his mathematical thinking more creatively (see Viitala, 2016b).

All in all, the results showed that even though the pupils seem similar on the surface level, on a closer look, they have very different skills and competences in mathematics. This is an indication that the framework allows different pupils to show different strengths, and also different weaknesses in problem solving and learning of mathematics. Hence, the framework could assist also teachers to pay attention to the aspects that pupils might need help with in developing their mathematical thinking, which in turn can help the pupils to recognise the knowledge, skills and affects that might need further developing (cf. FNBE, 2014, p. 377; Viitala, 2015b; see also Viitala, 2015a). An example of how teachers can use this framework to support their teaching is presented in Viitala (2015b).

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