Discursive shifts from school to university mathematics and lecturer assessment practices: Commognitive conflicts regarding variables

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We report part of an ongoing study which aims to characterise lecturers' assessment discourse, especially on closed-book examinations. We focus particularly on lecturers' discourses that concern the transition from school to university mathematics and we do so through highlighting one commognitive conflict regarding the use of variables in a task from an examination paper for a Year 1 course on Sets, Numbers and Probability offered in a UK mathematics department. We show evidence that the lecturer's assessment practices aim to facilitate students' avoidance of said conflict. Here, we focus on 6 out of 22 students' scripts which illustrate that, nonetheless, students still experience a commognitive conflict regarding the use of variables. More specifically, the students draw on the discourse of rationals instead of the discourse of integers when deciding the domain of the variables used in a Number Theory task.

Keywords: Undergraduate examinations, assessment routines, commognitive conflict, variables.

Introduction

Studies in mathematics education have focused on students' transition from secondary school to university (e.g. Gueudet, 2008). Part of how students experience said transition is evidenced in their engagement with examinations during their first year undergraduate courses. The nature of the tasks of these examinations has been studied using different theoretical frameworks (e.g. Tallman, Carlson, Bressoud & Pearson, 2016). Researchers have also examined lecturers' perspectives on examination tasks (Bergqvist, 2012; Tallman et al., 2016). In our study, we take a discursive approach in analysing examination tasks and lecturers' perspectives focusing on aspects of the transition from school to university mathematics. This theoretical approach allows a characterisation of the mathematical discourse the students engage in when solving the tasks and provides insight into lecturers' assessment practices and their expectations from students' responses.

In this paper, we analyse a task from a first year course on Sets, Numbers and Probability offered in a UK mathematics department. We build on previously reported work (Thoma & Nardi, 2016) in order to delve into lecturers' assessment practices facilitating students' transition to university mathematics in more detail. Specifically, we take the case of variables and the way these appear in a Number Theory task of the course's examination paper. In choosing this particular case, we take cue from previous works (e.g. Epp, 2011) which note that variables have diverse uses in mathematics, some of which often create difficulties for students' transition to algebra and other advanced topics. Of particular relevance here is the discussion by Biehler and Kempen (2013) about the difficulties with variables that students face.

In the part of our study reported here, we focus on a commognitive conflict that first year mathematics undergraduates experience with variables, when engaging in a Number Theory task and on their lecturer's assessment practices relating to assisting with this commognitive conflict. In what follows, we present briefly the theoretical framework of the study, the data and the examination task. We then analyse the task and the interview data with the lecturer who posed the task. Finally, we highlight the commognitive conflict regarding variables evidenced in the student scripts and conclude with a discussion of findings and how they are embedded into the larger study.

Commognitive conflicts and assessment routines facilitating discursive shifts

Sfard's (2008) theory of commognition is a discursive approach that is being increasingly used in mathematics education (Tabach & Nachlieli, 2016), as well as specifically in university mathematics education (Nardi, Ryve, Stadler & Viirman, 2014). Mathematics in this approach is a discourse that can be described in terms of the following four characteristics: *word use* (e.g. divisor), *visual mediators* (e.g. algebraic symbols), *endorsed narratives* (e.g. definitions) and *routines* (e.g. proving). The routines are distinguished in *deeds* ("an action resulting in a physical change in objects"; Sfard, 2008, p. 236), *rituals* ("creating and sustaining a bond with other people", p. 241) and *explorations* ("producing endorsed narratives", p. 259) with the explorations further categorised in *recall*, *substantiation* and *construction*. Of particular relevance to our analysis here is the construct of *commognitive conflict* "the phenomenon that occurs when seemingly conflicting narratives are originating from different discourses – from discourses that differ in their use of words, in the rules of substantiation, and so forth." (p. 257). For example, within the university discourse, variables can represent numbers coming from different domains. In a Number Theory context – where the domain of the variables is the domain of integers – a *commognitive conflict* may occur when students use variables in a ritualised way, influenced by the school discourse where the domain of the variables may not be made explicit. Consequently, the students may consider the variables in the Number Theory task as taking values from rational numbers. In addition to Sfard's original work, our analysis also draws on a framework – which uses the theory of commognition and social semiotics – that has emerged in the meantime out of a study that examines changes in the nature of UK secondary school finalists' (age 16) participation in the secondary mathematics examinations over the years 1987-2011 (Morgan & Sfard, 2016).

In our study, we examine students' participation in the university mathematics discourse taking also into account the lecturers' assessment discourses, particularly their rationale for the choices of the examination tasks and the wording of the tasks. Our previous analysis of examination tasks and lecturers' assessment practices (Thoma & Nardi, 2016) highlighted the following assessment *routines*: giving directions to the students regarding the steps their response to a task may take; structuring the tasks and subtasks in ways that allowed students to secure and optimise marks as they progressed from one part of a task to another; and, providing guidance regarding expected justifications in the students' responses. Overall, these routines aim at assisting students' shifting from school to university mathematics discourse. Here, we aim to extend our previous analyses, taking into account not only lecturers' assessment routines that facilitate shifts in the students' discourse, but also the commognitive conflicts that students experience when engaging with the tasks. We are, therefore, starting to look in tandem at aspects of students' experience (here: commognitive conflicts relating to variables in a Number Theory examination task) and lecturers' perspectives on – and intended practice relating to – this experience. In the following, we outline the larger study our paper originates in; and, introduce the examination task and a brief commognitive analysis of it. We then offer a brief analysis of the lecturer's perspectives on the task, highlighting those assessment routines that aim to help students avoid a commognitive conflict relating to variables. Finally, we present the students' scripts on this examination task and examine whether, and how, the commognitive conflict occurred.

The examination task and the participants of our study (lecturer and students)

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2.$$

(ii) (a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

Figure 1: Compulsory task on Sets, Numbers and Proofs

The data of our study consists of examination tasks from different courses, lecturers' interviews on those tasks and students' scripts corresponding to these examination tasks. The focus of this paper is on one task from the course Sets, Numbers and Probability. This is a first year course and has two parts: Sets, Numbers and Proofs taught in the autumn semester and Probability taught in the spring semester. The final examination includes six tasks: the first two are compulsory and the other four optional. One of the compulsory and two from the optional tasks are on Numbers, Sets and Proofs and the others on the Probability part of the course. At the final examination the students are asked to solve both the compulsory tasks and three from the optional tasks. The total grade of the examination is 100 marks and the pass grade is 40 marks. The focus of our paper is the compulsory task from the Sets, Numbers and Proofs part of the course (Figure 1). More specifically, in this part of the course, the topics covered are: Set Theory (notation, operations, cardinality and countability), Functions (introduction to functions, injection, surjection), Proofs (direct proof, proof by induction, proof by contradiction, proof by counterexample), Number theory (greatest common divisor, prime numbers, modular arithmetic) and Equivalence relations. The topic examined in this task is proof by induction and Number theory. Our analysis will focus on students' responses to the Number theory part of the task, task (ii). The model solution for part (ii) created by the lecturer for departmental use is in Figure 2. We note that this solution is not made available to the students.

(ii)(a) d is a *divisor* of a means that there is $k \in \mathbb{Z}$ with $a = kd$. [2 marks]

If d is a divisor of a and of b then there exist $k, l \in \mathbb{Z}$ with $a = kd$ and $b = ld$. Then for all $m, n \in \mathbb{Z}$ we have

$$ma + nb = m(kd) + n(ld) = (mk + nl)d.$$

As $(mk + nl) \in \mathbb{Z}$, it follows that d divides $ma + nb$. [2 marks]

(ii)(b) Following the method in lectures, let $a = 123$ and $b = 45$. Carrying out the Euclidean algorithm we obtain:

$$\begin{array}{rcr|rcl} a & = & 123 & & & \\ 2b & = & 90 & 45 & = & b \\ \hline a-2b & = & 33 & 33 & = & a-2b \\ 6b-2a & = & 24 & 12 & = & 3b-a \\ \hline 3a-8b & = & 9 & 9 & = & 3a-8b \\ & & 9 & 3 & = & 11b-4a \\ \hline & & 0 & & & \end{array}$$

We conclude that $\gcd(123, 45) = 3$ and that

$$3 = 11b - 4a = 11 \cdot 45 + (-4) \cdot 123 = 123m + 45n$$

where $m = -4$ and $n = 11$. [8 marks]

(ii)(c) No. Since 3 divides both 123 and 45 it follows from (ii)(a) that 3 divides $123s + 45t$ for all $s, t \in \mathbb{Z}$, but 3 does not divide 7. [2 marks]

Figure 2: Model solution of part (ii) of the compulsory task

There were 54 students who took part in the final examination. The marks ranged from 4 to 20, with the mean being 16.85 marks. The scripts of 22 students were selected by the first author to represent a variety of marks (Figure 3). The commognitive conflict regarding variables was observed in 6 students' scripts. Here we report: first, analysis from the lecturer interview data; then, a sample from the analysis of the students' scripts with a focus on the commognitive conflict regarding the role of variables in the task.

Task analysis and the lecturer interview

In part (i) of the task (Figure 1), the students are asked to engage in a substantiation routine (proof by induction). The wording of the task directs the students to this type of proof. In part (ii) the students are directed to engage first in a recall routine, giving the definition of a divisor, and then in a substantiation routine of a relationship describing the connection between the linear combination of *a* and *b* and the divisor *d* of *a* and *b* (iia). The students are then directed toward using the Euclidean Algorithm in (iib) and, in the last part (iic), they are expected to engage in a proof by contradiction (not explicitly mentioned in the wording of the task) in order to prove that the linear combination given is not divisible by 7 – see (Thoma & Nardi, 2016) for more detailed analysis of the task. For the purpose of this paper, we focus on the lecturer and student data corresponding to the second part of the task, (ii).

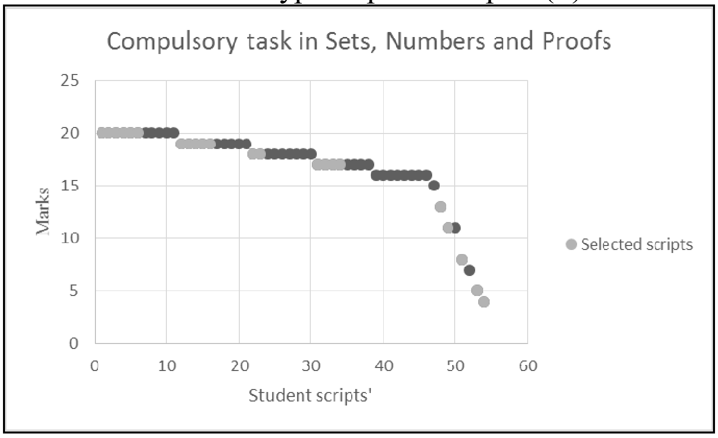

Figure 3: Marks from all the students' scripts

During the interview, the lecturer said:

> Lecturer: (…) my memory of school mathematics is that there was a lot of doing things but not necessarily a lot of formally defining things (…) And of course they came to university thinking that they knew what that [the definition of the divisor] meant but in this situation it really matters that they are restricting themselves to the-to the ring of integers (…) and all the symbols represent integers so what it means to divide is very different than if they were working with fractional numbers or something where they could write *a* over *b* and things like this.

Our commognitive analysis highlights the differences in the lecturer comments between the school discourse and the university discourse and, more specifically, with regard to the routine of defining and the importance of understanding that "all the symbols represent integers".

He highlights the differences between what the students are used to and what they are expected to do at university level. Our analysis sees this as the differences between the two discourses: on the one hand on the focus of the routines; on the other hand, on the constraints of the different discourses that exist within the mathematical discourse at university level. More specifically, the focus of the school discourse is on rituals (e.g. applying a well-rehearsed technique to reach a conclusion), rather than explorations (producing an endorsed narrative e.g. the definition of the divisor or the proof by contradiction). Also, the students, working with this definition have to restrict their work on integers – and not on rational or real numbers. The lecturer, then, speaks about the nature of the symbols involved. We recognize this comment by the lecturer as highlighting a commognitive conflict that students may face when defining the divisor as a rational instead of an integer, drawing on the discourse of rational numbers instead of the discourse of integers. Integer numbers are rational numbers, and making the distinction between the two – and then opting for working within the discourse of integers – is not something that these students have been routinely working with in school. In the excerpt that follows, the lecturer explains his assessment routines which aim to assist students with avoiding what we labelled as a commognitive conflict: subtask (ii) is gradually structured as first asking the definition, then, substantiating a narrative that draws on this definition, engaging with the Euclidean algorithm and, finally, combining all the above to engage in a proof by contradiction. He comments on the purpose of this gradual structure as follows:

> Lecturer: (…) what's being tested here is their ability to write down something formally and correct. And I would worry that if I didn't prompt them to write down formally the definition of what it means for one integer to divide another in the exam, in the pressure of the exam and so on, then their answers could start looking very 'creative' at the second part and they might start writing down fractions.

So, he aims that the gradual structure aids students towards achieving the expected solution. This can be thought of as a way of helping students avoid experiencing the commognitive conflict, where *a* and *b*, would be treated by students as rational numbers, instead of integers: this gradual structure serves as a reminder that they should restrict themselves in the discourse of integers. Additionally, the analysis shows that the lecturer stresses the routine of justification and the rigor of the university discourse compared to the school discourse, a further staple of the transition that these students are at the moment experiencing (Gueudet, 2008).

> Lecturer: (…) the only challenging part would be the last part, the part that requires some thought and they need to-to sort of understand or remember that somehow it relates to what happened up here [shows parts (iia) and (iib)] (…) to remind them that I want them to explain why they are answering what they are saying.

In the excerpt above the lecturer comments on the challenge of the (iic) part of the task and the purpose of the prompt "Explain your answer carefully". In this part of the task, the students have to engage in a substantiation routine which is based on the endorsed narratives that they have created for parts (iia) and (iib). The lecturer suspects that the students may omit justifying their response regarding the substantiation of the given relationship and aims that this prompt will help them do so.

From the above, we see that the lecturer has identified students' difficulties with the nature of the variables being used in this task. Our commognitive analysis sees this as evidence that the lecturer appears alerted to this difficulty as a difference between the school and the university discourse. The students, during their school years, gradually moved from the discourse of the natural numbers, to the one of the integers, then to the rational numbers and finally to the reals. Now, in this task, they are asked to endorse the discourse of the integers, which is subsumed in the discourse of rational numbers, within which they have been performing division of numbers in school. We now turn to students' responses which evidence that, despite aforementioned aid provided by the lecturer, experiencing this commognitive conflict was not avoided. Of the twenty-two student scripts analysed, six contained said evidence ([01], [03], [06], [11], [16], [17]).

The students' scripts

In student [01]'s response (Figure 4), we see that the symbolic visual mediator *m* belongs in the natural numbers, and not in the integers. The symbolic mediator *m* is given by the lecturer in the wording of the task and the domain that is assigned to this symbol are integers and not naturals. However, this student uses this symbol to indicate a new variable *m* which when multiplied with the divisor *d* would produce *a*.

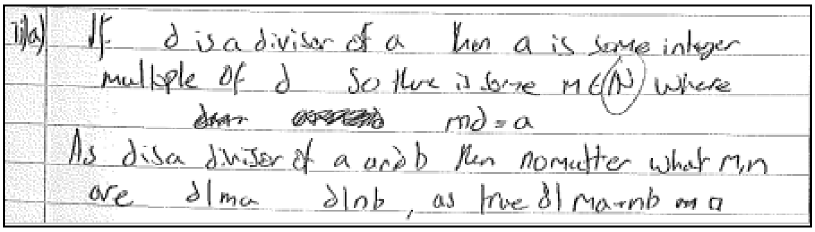

Figure 4: Student [01]'s script to (iia)

Then, in the next part, (iia), the student uses the symbol *m* to mean the *m* given by the wording of the task and states that "no matter what *m*, *n* are (...)" illustrating an unclear meaning making of the object divisor regarding the constraints in the discourse of integers. This student attempted to solve the next parts of the task by long division and then stopped. (We note that the circling around "ℕ", the symbol for the set of natural numbers in "*m* □ ℕ" originates in the script's marker).

Student [03] first communicates the relationship between the divisor *d* and *a* using written verbal visual mediators. In the second part of (iia) the student writes, using symbolic mediation, that *d* is a divisor of *a* and *d* is a divisor of *b*. However, in the symbolic realisation of the divisor, the student deploys fractions, with *d* being the numerator and *a* and *b* being the denominators. This signals that the student may see *m* and *n* as rational numbers, not integers, the set of numbers within which the task is set. Then, the student writes the relationship between the symbolic mediators *m*, *d* and *a* and concludes that *d* is equal to the product of *m* and *a.* The student used all the symbols given in the wording of the task to produce a narrative that involves fractions. Fractions are part of the discourse of rationals and this task asks the students to restrict their discourse to discourse of integers. We see the appearance of fractions in the students' writing as evidence of a commognitive conflict. Unclear meaning making regarding

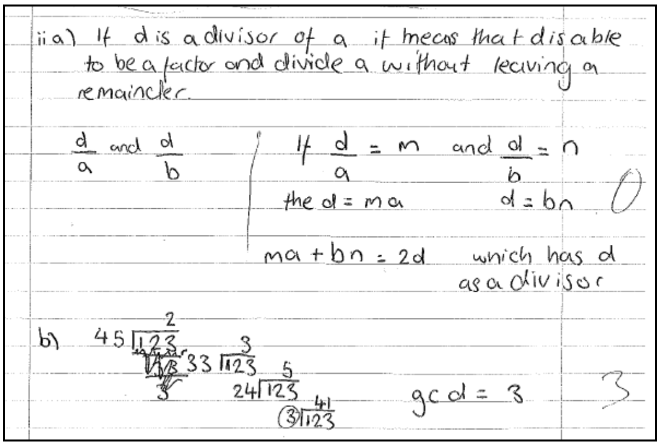

Figure 5: Student [03]'s script to (iia) and (iib)

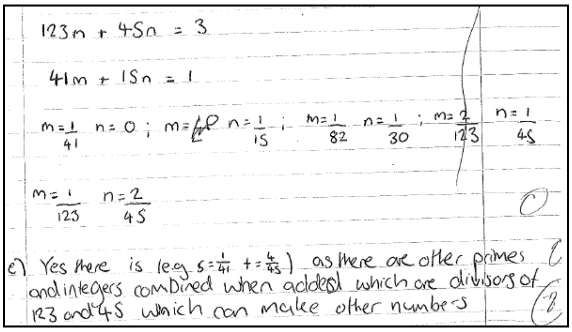

Figure 6: Student [03]'s script to (iib)-continued and (iic)

the object of a divisor is evidenced as the student starts by explaining that *d* is a factor of *a,* then engages in the discourse of rationals concluding that *d=ma* but then saying that the product 2*d* has *d* as a divisor. In figure 6, we can see the rest of the student [03]'s response. Having concluded that the greatest common divisor is 3 (Figure 5) using long division and not the Euclidean algorithm, the student writes 123*m* + 45*n* =3. Then s/he divides all the terms of the equality by 3 and takes different cases where the new equality is true.

In doing so though, [03] doesn't take into account the nature of the variable symbolic mediators and the variables become rational numbers. Also, there are multiple values in the rational numbers that satisfy this equality as can be seen in the response (Figure 6). Finally, in the last part of the task, the student responds affirmatively that there are integers *s* and *t*. However, the *s* and *t* s/he gives are rational numbers chosen to result in 7.

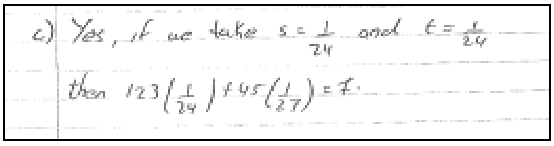

Figure 7: Student [16]'s to (iic)

Similar responses are given by two more students ([16] and [17]). Student [16] (Figure 7) does not give a definition of the divisor, attempts the substantiation of the relationship of the divisor, finds the greatest common divisor and uses only integers similarly in the identification of the integers *m* and *n* which give the linear combination of the greatest common divisor. However, when responding to (iic), instead of staying within the discourse of integers, [16] finds two non-integer rational numbers for *s* and *t* and confirms that the expression results in 7. Finally, in order to answer (iic), two students ([11], [06]) multiply the result they had found in (iib) with rational 7/3: this signals, again, the commognitive conflict regarding the nature of the variables *a*, *b*, *d*, *m*, *s* and *t* as the students have to restrict their discourse within the discourse of integers.

Symbolic visual mediation and the transition to university mathematics

Looking at the model solution produced by the lecturer (Figure 2) and the wording of the task (Figure 1), we can see that there are four different instances where the students have to define the symbolic mediators they use: first, in the definition of the divisor where an integer is introduced to illustrate the relationship between *a* and *d*; then, in the narrative which connects *a, b* and their divisor *d*; next, in the substantiation of the relationship between the linear combination of *a* and *b* and their divisor *d;* and, finally, in order to prove by contradiction that a linear combination of *a* and *b* is divisible by 7. In the last instance, the symbolic mediators on both sides of the equality have to be integers and, as 7 is not divisible by 3, the contradiction occurs. The wording of the task, where the lecturer stresses that all the variables in this part of the task are integers, and the structure of the task aim to restrict students' discourse within the discourse of integers. Our analysis suggests that lecturers design the tasks being aware of the potential commognitive conflict, adding to Bergqvist's (2012) results on what lecturers take into account when designing assessment tasks. The data from the students' scripts shows however that, in six out of the twenty-two analysed responses, students are experiencing a commognitive conflict that relates to making a distinction about the nature of the variables – a distinction which their prior (school) experience may not have prepared them for. We approach this issue therefore as a non-negligible aspect of the students' transition from school to university mathematics. Our results resonate with those in Biehler and Kempen's (2013) study. They found that, frequently, their participants would use symbols without providing information regarding the domain of the variable; in our case, not attending to such information results also in leaving parts of the task practically not answered – especially in the cases where the explicit request for "integers *s* and *t*" in (iic) receives non-integer responses.

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