Students' view of continuity – An empirical analysis of mental images and their usage

Erik Hanke[1] and Ingolf Schäfer[2]

[1]University of Bremen, Germany; erik.hanke@uni-bremen.de

[2]University of Bremen, Germany; ingolf.schaefer@uni-bremen.de

We investigate university students' mental images of continuity of real-valued functions by analyzing the answers of a questionnaire administered to Bachelor students at the University of Bremen. Our conception of mental images is based on concept images in the spirit of Tall and Vinner (1981) and the Grundvorstellungen (basic ideas) present in German subject-matter didactics (vom Hofe & Blum, 2016). For this purpose, we introduce the notion of "communicative simulacra." Furthermore, we catalog students' mental images of continuity that appear within this and preceding studies and demonstrate results on their acceptance in the study group. The used taxonomy and results are part of the first author's master's thesis (Hanke, 2016).

Keywords: Continuity, mental images, Grundvorstellungen, concept image, acceptance.

Introduction

More than 20 years ago Moore (1994) did an empirical analysis on the difficulty students face when they are required to give formal proofs. He identified among other factors that the students had little intuitive understanding of the concepts and their concept images were not adequate to do certain proofs. Moreover, Selden and Selden (2013) argue that the ability to choose the right conceptual representation is a vital part in proving and generally in problem-solving activities.

In the context of analysis the concept of continuity is one of the most fundamental notions needed to do rigorous proofs. It is well known that students have difficulties with the notion (Tall & Vinner, 1981). This paper focuses on the mental images that future math teachers, pure math and applied math students have, which mental images they find acceptable and which they can use to solve tasks.

Theoretical background

In the German tradition of subject-matter didactics the notion of *Grundvorstellungen* (regularly translated as "basic ideas") has gained much attention: The idea of Grundvorstellungen was extracted by vom Hofe (1995) after an analysis of related ideas of didactics of arithmetics and college-preparatory didactics by pointing out the importance of creating internal representations of mathematical notions in the learners' minds (vom Hofe & Blum, 2016). According to Kleine, Jordan and Harvey (2005) Grundvorstellungen link mathematics and reality by pointing out that modeling is a central mathematical process which fulfills requirements of mathematical literacy (application, structure and problem orientation): The authors argue that this is only possible after having acquired internal representations of mathematical concepts, so called Grundvorstellungen, which connect learners' experiences and mathematical knowledge with real life. Primary Grundvorstellungen are directly related to concrete objects and actions in the environment of the learners whereas secondary Grundvorstellungen consist of imaginative actions with mental representations (vom Hofe & Blum, 2016). The latter are particularly relevant for the notion of function and special classes thereof such as continuous real-valued functions.

We prefer to regard the essence of Grundvorstellung, using a subject-matter-didactical analysis, as a predominantly normative (or even prescriptive) approach to find internal representations learners should acquire in order to be able to recognize and use a mathematical notion in inner-mathematical or applied fields. But the idea of Grundvorstellungen is complemented by the wish of mathematical didactics specialists to observe actual mental models or images, respectively, that learners really develop (vom Hofe, 1995; vom Hofe & Blum, 2016; Kleine, Jordan, & Harvey, 2005).

The notions of *concept image* and *concept definition* by Tall and Vinner (1981) have been foundational for the existing literature on university students' conceptions of elements of analysis such as differentiation, integration but also limits and continuity (see Bressoud, Ghedamsi, Martinez-Luaces, & Törner (2016) for a recent discussion). The concept image comprises "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Besides, the concept definition is a form of words to specify the concept and to communicate it. It can be *personal* or *formal*, thus individually shaped or widely accepted by the mathematical community (Tall & Vinner, 1981). A formal concept definition rather reflects a normative viewpoint on what is actually forced to belong to the concept. We argue that concept definitions are part of the overall of concept images of a mathematical notion. Contrariwise, if learners are confronted with an existing concept definition they develop *concept definition images*, a part of their concept images that expressly comprises their associations with the definition. Additionally, it is understood that learners enter their acquisition process of a newly introduced concept with preexisting concept images (Tall & Vinner, 1981), and that teaching persons and environments can influence the acquisition of concept images (Bingolbali & Monaghan, 2008).

Since both the idea of Grundvorstellungen and concept images lack a distinctive description of what actually counts as internal representations, a conceptualization of *mental images* (Vorstellungen) was developed in (Hanke, 2016) which seems more appropriate to address the subtleties of precise and distinguished research questions in the scope of mental images. Mental images are substantiated as *individual* constructions and therefore *re*constructions of all kinds of mathematical notions. They are of *singular*, *regular* or *epistemological* nature, can be subdivided into *mental pictures* (Vorstellungsbilder) and *mental actions* (Vorstellungshandlungen) (Weber, 2007). Due to the premise of being able to be communicated mental images can be shared as well as accepted, rejected or even imposed on somebody.

The most important idea for our study—and in general empirical research—is the main observation that mental images cannot be observed. Thus, the only way to do empirical research about mental images is to study their *communicative simulacra*, the transformation of the inner world of a learner into observable entities such as spoken words, written solutions to exercises and so on (Hanke, 2016). In particular, communicative simulacra do not reflect normative assumptions on a notion as it is the case with Grundvorstellungen. In case of answers to a direct question on mental images (e.g. "What is *your* intuitive meaning of continuity?") we will speak of *verbalized simulacra*. With this terminology we reflect upon the fact that what is actually communicated by a learner depends on the occasion of communication, not necessarily needs to be the full entity of associations the learner has to the notion in question, nor can we be sure that the learner is aware of what could be the objective of the researcher. Rather we find blurrings of the actual mental images of learners that could potentially be sharpened by further qualitative analysis. In particular, verbalized simulacra are shaped by the learner's understanding of the concept in question and are only a subset of communicative simulacra which, in turn, can be expressed by different forms of communication. Here we concentrate on descriptions of communicative simulacra.

According to Moore (1994) and Selden and Selden (2013) we believe that the mere knowledge of definitions, the ability to reproduce them or the setup of mental images for a mathematical concept do not necessarily mean that the students are able to use the concept. Also, we believe that the more mental images students have the more they are able to apply at least some of these in inner-mathematical situations or in contexts. Moore's (1994) term *concept usage* is related to our idea of distinguishing between verbalized simulacra of mental images and the usage of (probably different) mental images as required in the third section of our questionnaire. The second part of the questionnaire provides insight in the *acceptance of attitude* (Einstellungsakzeptanz) and *acceptance of usage* (Nutzungsakzeptanz) (Weber, 2007) (see next section).

The review of central papers (Bezuidenhout, 2001; Núñez & Lakoff, 1998; Schäfer, 2011; Takači, Pešić, & Tatar, 2006; Tall & Vinner, 1981) on concept images and related results on students' conceptions of continuity lead to the classification in Table 1 of the eight possible mental images that are reported in the literature following Mayring's (2015) methodology of *qualitative content analysis*. We emphasize that these categories are representatives of communicative simulacra identified in the literature and we do not intend to judge about their formal or normative correctness.

Connections to the concept of integration is hardly ever noticed explicitly and therefore in case of appearance subsumed under miscellaneous. Likewise, the concept image of "pulling flat" the graph of a real-valued continuous function (Tall, 2009, p. 487) could not be found in any of the students' responses. It seems to be related to the rubber band metaphor often used in topology and usually is not part of standard German textbooks or lectures on analysis in one variable. Additionally, Schäfer's (2011) Grundvorstellungen for real-valued functions (controlled stability while wiggling at a point, possibility of approximation at a point and connectedness of the graph) are subsumed in the categories of Table 1.

#	**Category**	**Example**
I	Look of the graph of the function	"A graph of a continuous function must be connected"
II	Limits and approximation	"The left hand side and right hand side limit at each point must be equal"
III	Controlled wiggling	"If you wiggle a bit in x, the values will only wiggle a bit, too"
IV	Connection to differentiability	"Each continuous function is differentiable"

V	General properties of functions	"A continuous function is given by one term and not defined piecewise"
VI	Everyday language	"The function continues at each point and does not stop"
VII	Reference to a formal definition	"I have to check whether the definition of continuity applies at each point"
VIII	Miscellaneous	

Table 1: Categories for mental images of continuity

Setup of the study and methodology

Our research questions have been:

1.) What mental images do students express by verbalized simulacra?

2.) What mental images do students accept and make use of in argumentation?

3.) Is there a difference between students who want to become teachers and those studying pure and applied mathematics with regard to mental images or concept usage?

We distributed a questionnaire to 54 Bachelor students (first-year pure and applied mathematics and second-year mathematics teacher students) in Bremen after the completion of a lecture with exercise classes on Analysis I (Hanke, 2016). The chosen methodology of the questionnaires is very similar to the one used often to investigate concept images (e.g. Tall & Vinner, 1981; Bezuidenhout, 2001; Nordlander & Nordlander, 2012). Our questionnaire, described in detail below, is an extended version of those described in Tall and Vinner (1981) and in particular Schäfer (2011). No questions concerning applications are given in order to identify students' conceptions of continuity solely related to the mathematics itself. New is the differentiation as described in the taxonomy of communicative simulacra and the comparative approach of acceptance of attitude and acceptance of usage, i.e. if students accept certain concept images and if they can apply those different images to some example functions.

In the first part of the questionnaire the students were asked to freely verbalize what the intuitive meaning of continuity from their point of view is. In the second part we probed the acceptance of attitude of the following verbalizations of mental images presented to the participants in fictive statements on a 6-point Likert scale (totally decline (0), …, totally accept (5)):

#	**Description**
1	Having minima and maxima is characteristic for continuity
2	Limit definition of continuity
3	Weierstraß ε-δ-definition / preimages of small open intervals contain small open intervals
4	Graph has no holes
5	Controlled wiggling

6	Connection to differentiability ("a function is not continuous at a point if it cannot be differentiated at that point")
7	Graph has no jumps
8	Graph does not swing too much back and forth

Table 2: Mental images of continuity probed in the second part of the questionnaire

Furthermore, the third part of the questionnaire focused on acceptance of usage of mental images since we asked to give arguments for whether the following functions in Table 3 are continuous at the respective locations with multiple mental images.

1.) $f: \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 0 \text{ for } x \in \mathbb{R} \setminus \mathbb{Q} \\ x \text{ otherwise} \end{cases}$ at $x = 0$	2.) $f: \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 0 \text{ for } x \in \mathbb{R} \setminus \mathbb{Q} \\ x \text{ otherwise} \end{cases}$ at $x = 1$
3.) $g: \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 \text{ for } x < 0 \\ 0 \text{ otherwise} \end{cases}$ at $x = 0$	4.) $h: \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ at $x = 0$

Table 3: Different functions in questionnaire

Summarizing, we are interested in the threefold of mental images through communicative simulacra: verbalized simulacra and the usage and acceptance of mental images observable in communicated outcomes. Due to page restrictions, we limit ourselves on an overview and provide some statistics.

Results and discussion

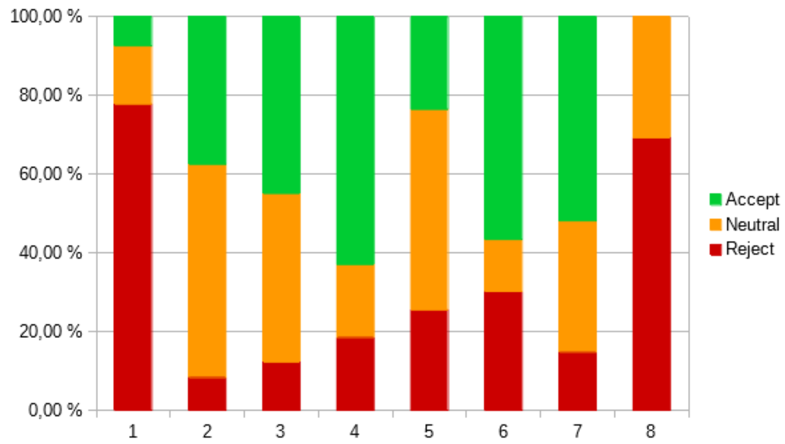

Figure 1: Acceptance of item on Likert scale from 0 to 5: Reject (0, 1), Neutral (2, 3) and Accept (4, 5) in percent of answers

All answers to our questionnaires were categorized according to Table 1. Multiple responses of the students were not only possible but desired and multiple categorization of a single answer into the categories was also possible. The categorization of the answers to the first question "What is the intuitive meaning of continuity from your point of view?"—i.e. the verbalized simulacra of mental images of continuity—led to the following: 5 students did not give an answer, 37 answers fell into only one category, 11 in two and the one remaining answer in three categories. Around 70% of the overall codes were found in “look of the graph“ (I) and all the other categories appeared in no more than 10% of the cases each.

The “look of the graph” (I) is the dominant mental image among students when asked to give one. Nevertheless, some students are able to accept other mental images as well. Figure 1 illustrates this. While the items “graph has no holes” (4) and “graph has no jumps” (7) are accepted by the majority, items close to the limit (2) or Weierstraß definition (3) of continuity have 40% to 50% acceptance. That there seems to be a majority who wrongly connects differentiability as necessary condition for continuity may be a problem of the item which included two negations. Based on the very high rejection rates for the items „Having minima and maxima is characteristic for continuity“ (1) and „Graph does not swing too much back and forth“ (8) in the second part of the questionnaire, it seems to be certain that these are not common misconceptions about continuous functions.

The functions *f*, *g* and *h* in Table 3 were all known to the students and had been part of the course in analysis and also of the exercises. In contrast to Tall and Vinner (1981) we did not give a picture of the graphs. The functions *f* and *g* seem familiar to the majority of students so about 50% are able to give a correct answer. The function *h* seems more complicated and most students do not answer the question at all and about half of the answers are false. This item is one where there is a real difference between those who study to become teacher and those who want to work as mathematicians. In the latter group the percentage of a correct answers is about twice as high (Hanke, 2016).

To identify group differences between the different study groups (pure and applied math students vs. future teachers), we counted the occurrences of every category in Table 1 in the answers of the students for each of the questions. Concerning the overall usage of certain mental images measured with the coding of all answers to all functions of the third part of the questionnaire Fisher’s test on the resulting contingency tables did not yield a significant result. We interpret that there are no observable differences in the acceptance of usage of the different study groups. Using the Kruskal-Wallis-Test, we could not find statistically significant differences except for the acceptance of the limit definition (2) in part two of the questionnaire ($p < 0{,}03$) where teachers students tended to express their acceptance with higher values on the Likert scale than the others.

Comparing the results with Schäfer (2011) we could identify more detailed concept images via their verbalized simulacra (categories 4 to 7). While the concept image of “look of the graph” (I) was dominant here as well, it is not so dominant as in (Schäfer, 2011). We see a more diverse pattern in the argumentation for the three functions instead.

The most interesting part of the empirical results is that the same mental image is used by the students either to justify a wrong or a correct answer (based on their judgment whether the given function is continuous or not; cf. Figure 2): The look of the graph (I) was used most frequently for a correct but also a wrong answer. For example, the graph of function *h* is connected (i.e. has no jumps) but the function is discontinuous at the origin. Among the answers to this function we found e.g. “Yes [*h* is continuous], since [it is] going through, without gaps or jumps,” or “The function looks continuous, since it does not ‘jump’.”

General properties of functions (V) were even used more often for a wrong than for a correct judgment of continuity. Again, for the function *h*, some students argued it is "discontinuous because of a pole," but also *h* is "continuous at *x=0*, since it lies in the domain of the function." For this function we could also find various other justifications for (dis-) continuity like "it gets area-like at the origin" or "it wiggles too much."

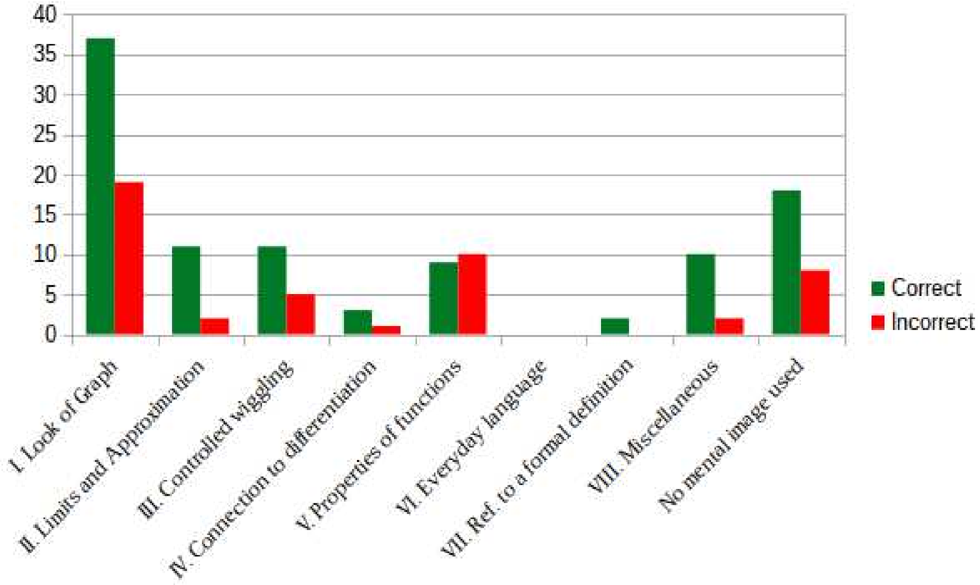

Figure 2: Numbers of correct and incorrect answers per category summed up for all four functions

Outlook

In this study we provided a taxonomy that guides empirical research related to mental images of mathematics students in several directions. We pointed out that communicative simulacra of mental images of real-valued continuous functions depend on the context in which mental images are used: Figures 1 and 2 show that the spectrum of mental images used or accepted is broader than the spectrum of explicit verbalized simulacra. We also found out that in the overall of justifications of (dis-) continuity a single mental image does not exclusively help or misguide. A first step into mental images of metric space-valued continuity is also given in (Hanke, 2016).

We believe that future research on teachers', doctoral students', tutors' or university lecturers' conceptions of continuity will provide insight into similarities and differences between social groups in the overall process of teaching and learning of a particular mathematical notion. This will be of particular importance for the teaching of real-valued continuity in today's university classrooms. Since continuity is disappearing from the curricula in secondary schools in Germany, it would also be interesting to find out more about teachers' judgments of the adequacy of teaching continuity in schools as a prerequisite for important facts on differentiability and integration such as the fundamental theorem of calculus.

Acknowledgment

We would like to thank the Bundesministerium für Bildung und Forschung for its support in the subproject "Spotlights Lehre" within the project "Schnittstellen gestalten."

References

Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *Int. Journal of Math. Educ. in Science and Technology, 32*(4), 487–500. doi:10.1080/00207390010022590

Bingolbali, E. & Monaghan, J. (2008). Concept image revisited. *Educ. Stud. Math. 68*(1), 19–35. doi:10.1007/s10649-007-9112-2

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and learning of calculus* (ICME-13 Topical Surveys). SpringerOpen. doi:10.1007/978-3-319-32975-8

Hanke, E. (2016). *Ausprägung und Akzeptanz von Vorstellungen zur Stetigkeit bei Mathematikstudierenden* (unpublished master's thesis). University of Bremen, Bremen, Germany.

vom Hofe, R. (1995). G*rundvorstellungen mathematischer Inhalte*. Heidelberg: Spektrum.

vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a category of subject-matter didactics. *Journal für Mathematik-Didaktik, 37*(1), 225–254. doi:10.1007/s13138-016-0107-3

Kleine, M., Jordan, A., & Harvey, E. (2005). With a focus on 'Grundvorstellungen' Part 1: a theoretical integration into current concepts. *ZDM, 37*(3), 226–233. doi:10.1007/s11858-005-0013-5

Mayring, P. (2015). *Qualitative Inhaltsanalyse: Grundlagen und Techniken* (12th ed.). Weinheim, Basel: Beltz.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249–266. doi:10.1007/BF01273731

Nordlander, M. C., & Nordlander, E. (2012). On the concept image of complex numbers. *Int. Journal of Math. Educ. in Science and Technology 43*(5), 627–641.

Núñez, R. E., & Lakoff, G. (1998). What did Weierstrass really define? The cognitive structure of natural and ε-δ continuity. *Mathematical Cognition, 4*(2), 85–101.

Schäfer, I. (2011). Vorstellung von Mathematiklehramtsstudieren[!] zur Stetigkeit. *Beiträge zum Mathematikunterricht 2011, 2,* 723–726. doi:10.17877/DE290R-7933

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *TME, 10*(1/2), 303.

Takači, D., Pešić, D., & Tatar, J. (2006). On the continuity of functions. *International Journal of Mathematical Education in Science and Technology, 37*(7), 783-791.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Tall, D. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM, 41*(4), 481–492. doi:10.1007/BF00305619

Weber, C. (2007). *Mathematische Vorstellungen bilden. Praxis und Theorie von Vorstellungsübungen im Mathematikunterricht der Sekundarstufe II.* Bern: hep Verlag AG.