Q² a game used in a task design of the double quantification

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This article deals with student's interpretation of multi-quantified statements. The efficient interpretation of such statements is very important, for example, for those who try to understand and use the formalism of the definition of limit of a function. Literature points some students' difficulties in the interpretation of double quantified statements. I am using the scientific debate framework to design and implement tasks by means of two fundamental steps. In the first step, I draw on the results of a questionnaire to identify and determine these difficulties. The aim of the second step is to design the targeted tasks by considering the results of the first step; these tasks are based on a game called Q². The implementation of the design shows that students understand that there are two kinds of interpretations, and that conventions are needed.

Keywords: formalism, quantification, limit, scientific debate in class, game Q².

Quantified statements problem

The mathematics formalism uses massively quantification and specifically multi-quantification. The research shows that students have difficulties with the interpretation of multi-quantified statements (Dubinsky & Yiparaki 2000, Chellougui 2009, Piatek-Jimenez 2010). EA statements corresponding to "Exists…for all…" sentences must be distinguished from AE ones corresponding to "For all…there exists…". Dubinsky & Yiparaki (2000) study the impact of two main variables on the interpretation of double quantified statements. The first variable is the order of quantifiers (AE or EA) and the second one is the kind of statement, mathematical or non-mathematical. They show that the interpretation of non-mathematical statements is essentially correct but the interpretation of mathematical ones is difficult for students. For AE mathematical statements, the students' interpretation is quite efficient, whereas their interpretation of EA ones seems to be done through an inversion: the EA statement is very often interpreted as the AE corresponding statement. Moreover, it is noticed that "the students did not appear to care of the syntax of a statement to analyze it […] the student did not appear to be aware of having engaged in interpreting the questionnaire statements." (Dubinsky & Yiparaki, 2000, p. 53). Barrier (2009) questions these results, showing that even some AE statements can lead to difficulties of interpretation. The inversion of interpretation of an EA statement is also noticed by Chellougui (2009) when she asks students to define the upper bound M of a set A and when they almost all answer by what she calls "a strange definition": « $\forall x \in A, \exists M \in \mathrm{IR}, x \leq M$ ». Piatek-Jimenez (2010) confirms the asymmetric perception and interpretation of those two kinds of double quantification in the mathematical field. Barrier (2009) shows that some AE statements, even non-mathematical ones, can also lead to difficulties of interpretation for students. In the specific field of the study of understanding the definition of the limit, the same kind of difficulties are also often mentioned (Swinyard 2011; Oehrtman & al. 2014).

To overcome these difficulties Dubinsky & Yiparaki (2000) have presented a game based on the dialogical logic (explained below) and used with the students to make them aware that two kinds of interpretation can be used and that this depends on rules of interpretation linked to the places of quantifiers. The results have shown that this seems to help students to understand such statements but in some cases it does not, and in a few cases this has created more confusion.

I want to pursue these researches by designing tasks to overcome the difficulties of the interpretation of double quantified statement using the frame of scientific debate. This implies two related goals. The first one deals with the identification of the difficulties of this interpretation. I specifically explore the role of some other potential variables: the set of quantification (familiar or not, finite or infinite), the semiotic representation of the quantified variables (formal or usual language) and the kind of relation involved in the predicate (familiar or not). The second goal is to design and to implement tasks by considering the results of the study related to the first goal. A new game called Q^2 based on the interpretation of double quantified statement to determine a winning strategy will play a fundamental role in this design.

The aim of this paper is to study the following question: is there a domain of values of variables where students are giving a right interpretation and if yes how to take advantage of this successful domain to enlarge it in order to mobilize this knowledge, for instance in the understanding of the ε-δ definition of limit? Specifically, how to create the students' need of the rules for the interpretation?

The framework of scientific debate: a tool for designing tasks

The scientific debate is a socio-constructivist approach of learning and teaching mathematics based on two main principles: 1) the need of a new knowledge emerges when one realizes that his previous conceptions may lead to contradiction; 2) the organization of appropriate debates among students that are collectively seeking truth permits firstly to express and share previous conceptions about the targeted subject, secondly to encounter the limits of these conceptions, and thirdly to be able to understand the institutionalisation related to this knowledge that is made by the teacher. Three fundamental steps shape the design of tasks by means of the scientific debate framework:

1) The first step consists on epistemological and cognitive studies of the targeted mathematical concept: here, the role of quantifiers in formal statements. This should lead to identify some main meanings of the concept: in this case, the need of convention of interpretation to have a common interpretation. And this should also lead to identify the breaking point of conceptions relative to this meaning, the fact that the lack of conventions can become crucial to students who realize that many interpretations are possible.

2) The second step concerns the design of the tasks, which is done by using mainly two kinds of questions to initiate the debate. The first kind of question concerns the truth of a conjecture: is it true or false? This conjecture can be given by the teacher or can come from the student after a call for conjecture. The second kind of question concerns the nature of an object for a conjecture or a property: is it an example, a counter example, or off topic(neither example nor counter-example) for this conjecture, and is it an example for a property? The example can come from the teacher or for a call for an example. A vote is made: do you think it is true, false or *something else* (an example, a counter example, off topic or *something else*)? The possibility of voting for *something else* is given to preserve the authenticity of the other votes. Then a debate is organized by the teacher between

those who have different view-points. The teacher never gives any own clue about what is debated but tries to maintain a level of interaction by emphasizing the contradictions between students.

3) The third step concerns the level of experimentation and its analysis: what actually happens is confronted to what was expected to happen. Specifically, this analysis leads to discuss the efficiency of the choices made in the two first steps.

The whole study is conducted according to the aforementioned three steps. In this paper, I show my findings from the study related to the first step and I give more details about the elaboration and the results of the two final steps. For the first step, I have chosen to use a questionnaire on a sample of 181 students in their last year of secondary school to identify students' difficulties with the interpretation of double quantified statements. For the second step, the results of this questionnaire are used in a way that is aiming to make students aware that the lack of rule of interpretation is not a problem in a certain domain but leads them to conflicts out of this domain. This design is based on a game Q^2 in which the interpretation of double quantified statement is crucial, the question that will initiate the debate concerns the way to win this game. For the third step, I experiment this design for students in their last year of secondary school in a scientific class composed of 29 students.

Task design

Background: The result of the questionnaire

I mainly studied five variables in the interpretation of double quantified statements: place of quantifiers, kind of statements (mathematical or not), the set of quantification (familiar or not, finite or infinite), the semiotic representation of the variables (formal or informal language) and the kind of relation involved (familiar or not). The first finding is that the rule of "correct interpretation for AE statements and inversed one for EA statements" seems not so obvious: some EA statements are perfectly interpreted whereas some AE statements are interpreted through an inversion. The second finding is that some other variables play also an important role in the interpretation (e.g. width of the quantified set, formalization of variables,..). This second finding is also that there exists a domain of correct interpretation for both EA and AE. This domain is made of non-mathematical statements quantified on a "little" finite set (less than ten values) without any formal variable and with a familiar relation (for more details see Lecorre 2016 a). These findings are then used to design the tasks aiming the interpretation of double quantified statements.

The game Q^2

The game presented by Dubinsky and al. (2000) is based on the dialogical logic of Lorenzen (1967) which gives a way to decide of the truth of quantified statements using a codified dialog between a proponent and an opponent. In this game, for example, if the sentence is "for All x there Exists y such that $R(x;y)$", the A-player chooses x and the E-player has to find a y such that $R(x;y)$ is verified. If he fails, the A-player wins, otherwise the A-player can give another x (same kind of rule for EA). I call this game a direct game whereas the Q^2 game is an inverted game: the players are constructing the set of quantification to make a statement true or false.

For the Q^2 game I choose values of the variables that make it an easy game to play: non mathematical field, small set of quantification, familiar relation. This choice is made to permit students to get into the game and into the interpretation of associated statements.

Let's take an example with the statement S: "For all red letters, there exists the same black letter" and the given grid:

H	U	H
L	U	L
U	H	T

Figure 3 : a grid for the game Q^2

This is a two players' game: the red player who has a red pencil and the black who has a black one. Let's say that the red player has to start (starting rule) and plays for the falsification of the sentence (winning rule). He has, first, to circle one letter with his red pencil. Then the black player circles one letter and so on until all the letters have been circled. With the given rules, if the statement S is false then the red player wins, but if it's true the black one wins. This game has, of course, many variants, beginning with the filling of the grid and the starting and winning rules. This game is the heart of a design which aims precisely to enlarge the domain of good interpretation; I am going to see that a smart use of Q^2 in the design has the potential to reach such a goal by emphasizing the lack of convention of interpretation.

"The Q^2 situation"

The principle of the Q^2 situation is to provoke conflicts of interpretations that will lead students the need of convention of interpretation. The situation Q^2 is divided into four periods:

- The first period aims an appropriation of the game Q^2 by playing.
- The second period aims the concept of winning grid for the game Q^2.
- The third period aims a conflict of interpretation, in a way that students feel the need of conventions of interpretation. At that stage, the conventions are given.
- The fourth period is just an application of these conventions on the unsuccessful domain where the values of the variables lead students to difficulties of interpretation.

In the first period, a paper is given that contains eight games of Q^2 to play (each game is defined by a statement, a starting rule and a grid). These games are designed for the two players to have opportunities to win and to start to have ideas on how to play to have good chances of winning. In fact, with such a game, with "good players", the winner depends only on the statement (EA or AE), the starting rule, and the grid given.

The second period aims the definition of winning grid. The students are asked to give the winning grids for a given rule, then a debate is organized about these propositions: are they winning grids, or not? The contradictory positions about the propositions should lead students to identify the lack of a definition of winning grid. A winning grid is, in fact, quite difficult to define in a mathematical way for this level of student (double recursive definition). Here, a definition such as "a grid is a winning grid for red if when red plays "cleverly", he is sure to win, even if black also plays cleverly" is largely sufficient for this design. When the students show a need for a definition, the above definition is given.

The third period aims to highlight the lack of convention of interpretation. Once again, for a given rule, the students are asked to give the winning grids. There should be no more conflicts about what is a winning grid in general, but new conflicts should appear about the propositions: is this grid really a winning grid with this rule? This should happen because the winning grids depend on the interpretation of the rule which is a double quantified statement. And the need of a convention should appear with the impossibility to find a common agreement (is it a winning grid or not?). The didactical principle which is used here is the following one: it seems very difficult to organize a direct confrontation of the different rules used by students to interpret a double quantified statement, because this problem, taken as a general one, is too theoretical and depends on so many variables (findings of the questionnaire). On the contrary, it is very easy to create a conflict on concrete consequences of the interpretation of such statements. Here, the conflict holds about the question"is the grid a winning one or not?" Then, trying to understand each other, and trying to convince their peers, students are going to explain their own interpretation. And then it will appear that the implicit conventions used by students are contradictory. Students can realize that without common conventions, no agreement is possible. Then these conventions are given in the manner of the dialogic logic (Lorezen 1967). At this stage, the game Q^2 plays as a preparation to such rules by simulating a game between a proponent and an opponent. The design here aims much more the awareness of the need of convention than the "right rules". The aim is to make students aware of the necessity to check the validity of their interpretation relatively to the adopted conventions.

The fourth period consists in verifying, still using debates that the given rules can lead to agreements, and even that in unsuccessful domain of interpretation: the rules are helpful, efficient and change this domain into a successful one.

Results

The first period of the situation Q^2 (playing the game) shows a good appropriation of the game: the winning grids are almost always won by the one for who the grid is a winning one, which means that the students are playing "cleverly". Some strategies seem to begin to be used. And, above all, almost all the decisions about who is the winner are correct. All this is coherent with the results from the questionnaire in terms of successful domain of interpretation.

The second period lead, as expected, to the question of the definition of a winning grid. The definition is given.

The third period begins, unexpectedly, as soon as the definition of a winning grid is given: the question of the interpretation is raised by students in a debate. I present some extract of the script of this debate to explain this unexpected acceleration. Six winning grids had been given by the students in the second period for the game S1:"There exists a red square such that all black square have the same symbol" where red starts and plays for true:

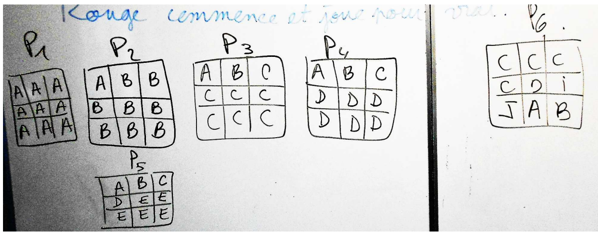

Figure 4: the winning grids for red given by the students

The grid P1 is put into debate and everyone agrees that it is a winning grid for red. Then P2 is put into debate (28 votes for a winning grid for red and 1 for “something else”). Loïc who has voted something else changes his mind and explains why, for him, it is a winning grid for red:

Loïc: Because red start and as he plays to win he takes the square A and…

Teacher: you’re saying that “red plays A first” yes and what?

Loïc: Then black takes only B squares.

Teacher: Black only takes B squares. Why, in the end, red wins?

Loïc: Because….

Hadrien: Because Black has only B squares.

But Quentin disagrees with this explanation:

Quentin: And because red has it also (One B square)

But Hadrien and Louis do not agree with this addition:

Hadrien: no, he has A squares and B squares.

Quentin: yes!

Louis: I think that I should just say that there exists a red square.

The sequence above shows that these students do not need to disagree on the fact that it is a winning grid or not to begin to explain their own interpretation: “the same symbol as the red square” (Quentin), or “the same symbol for all black squares” (Loïc, Hadrien). Then Fabio, in the same way, explains that nothing must be added in contrary of the sayings of Quentin:

Fabio: I do think that what added Quentin is not necessary.

Then Quentin will propose a grid to strengthen the differences of interpretations:

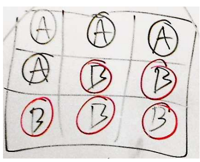

Figure 5: the grid of Quentin (As in black and Bs in red)

Leaving aside for a while the problem of the winning grid, the teacher asks whether the statement S1 is true or not, according to the grid of Quentin. Twelve students are thinking that the statement is true, ten false, while six students vote something else. After a long debate, some conclusions are raised by Fabio, Mickaël and Louis:

> Fabio: There are some, like me, that can think that all the black squares have got the same symbol is enough and some other, like Sébastien, who are thinking that there must be a red square that has got exactly the same symbol as any black square.
>
> Mickaël: I'm asking: for who thinks that only black squares have the same symbol is enough, what is this red square that exists?

This makes Hadrien change his opinion, but Louis does not agree with this change:

> Louis: There are two opposite opinions just because we're not thinking the same.
>
> Teacher: Ok, you are not reading the same way…That is what Fabio said…
>
> Louis: exactly!

Juliette and Maxime then explain why they voted something else:

> Maxime: That is exactly why, from the beginning, I voted Other.
>
> Juliette: So do I.

For students, the need of a convention is sufficiently clear for the conventions of interpretation to be given. These conventions are then used successfully for the applications of the fourth period, even for the most unsuccessful domain of interpretation (mathematical statement with formal variables and infinite sets of quantification). I can explain the runaway of second period by two main reasons. The first one is that the successful domain of interpretation is effectively successful but seems also very fragile. The students have, most of all, correct interpretations on this domain, but they seem to have no real conventions to refer to and only personal conviction. And whenever a different interpretation from theirs is produced, the students do not know anymore what to think about this new interpretation and therefore about their own one. The second reason is that the use of scientific debate, thanks to its focus on the search of truth, lead students to face contradictions and to try to solve them. Here, those students seem to have a kind of intolerance for contradiction: they do agree on the fact that P2 is a winning grid, but do not agree on the reasons that found it. This leads them to get aware that different rules are used to interpret a double quantified statement and that, without a convention, it is impossible to find an agreement.

Conclusion

The Q^2 situation, built on the scientific debate framework, achieved to make the interpretation of the double quantified statement the main object of students' discussion. The contradictions that the lack of conventions of interpretation necessarily implies are then emerging in that discussion. This lack which was invisible to students is suddenly highlighted by these contradictions.

More precisely, the questionnaire led to identify some variables playing a role in the interpretation of students (kind of set of quantification, kind of semiotic representation of the quantified variables and the kind of relation involved). There is a domain of the value of those variables which gives a good interpretation: non-mathematic statements quantified on a small set without any formal variable and with a familiar relation. This successful domain is used to shape the game Q², to make it easy to play, so that the students can then discuss about the specific winning cases. Different ways of interpreting become central in the discussion. This Q² situation led students to be ready to receive the conventions of interpretation of double quantified statements. Indeed, they have experienced the need of these conventions. I can say that this is the beginning of the learning of the double quantification, insofar as the validation using variables defined in function of the other variable, remains to be learned. The described situation Q² mainly aims that there are two kinds of double quantifications that should be differenced according to the order of the quantifiers in the statement and the convention of interpretation of the predicate. One month later, the same students had to face double quantified statements in a situation aiming the definition of limit. They experienced, once again, the need of conventions when they encountered another conflict of interpretation of these statements, so they checked the conventions to decide on their own (Lecorre 2016 b).

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