Introduction to equation solving for a new generation of algebra learners

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Understanding of equality and solving equations are some of the big ideas in algebra. They have been in focus in early algebra research for some decades and in many countries it is now usual to work with equalities and solving equations using informal methods from early school years. However, it is not clear how the transition to formal methods of equation solving could be conducted in a best way in order to maintain students’ interest and enhance their understanding of the need for more powerful methods. We shed light on the issue by reporting on what happens when three teachers introduce linear equation solving in Grade 6 (age of 12) in Finland. We especially consider how the introduction could support students’ development of an algebraic understanding of equality and their engagement in more formal mathematics.

Keywords: Equations, elementary school mathematics, equality, algebra

Introduction

Learning to solve linear equations in a more formal manner may be a critical moment in the school mathematical experiences of a student. The student is perhaps able to figure out the missing value in an equation written as an open number sentence but does not understand how and why the standard algorithm works for solving the very same equation. This particular situation is shown to be associated with how the student understands equality (Kieran, 1981; Knuth, et. al., 2006, 2011; Vieira, Gimenez & Palhares, 2013). In arithmetic it is often enough to understand the equal sign as an operator, as a do something -signal (Kieran, 1981). In algebra, however, the student should understand equality between two expressions as an equivalence relation that doesn’t change. Such algebraic understanding is not supported if the student is taught to memorize procedures as ‘move terms from one side to the other side of the equal sign and change the corresponding signs’ or ‘do the same thing on both sides’. Successful equation solving is connected to a relational meaning of the equal sign and understanding the notion of equation as a statement about an equivalence structure (Knuth, et. al., 2006, 2011; Stacey & MacGregor, 2000). Further, although students now often learn to consider the equal sign as a relation and solve simple equations with informal methods already in primary level, it is not sure that they can operationalize the meaning of their experiences when moving to more formal methods of equation solving. For example in the Finnish curriculum early algebra has been considered for several decades while the introduction and the working manners in algebra in secondary level have not been changed accordingly. Scholars have only recently started to theorize about possible learning progressions with respect to algebra in different school years (Cai & Knuth, 2011).
The general aim of this paper is to shed light on the issue of the transition from informal to formal methods by analyzing how students in grade 6 classrooms in Finland are introduced to solving linear equations with one unknown.

**Learning to solve linear equations**

A distinction can be made between an arithmetical and an algebraic notion of equality and a corresponding difference in arithmetical and algebraic understanding. Following Filloy and Rojano (1986) and Vlassis (2002), if the unknown in a linear equation appears only on one side of the equal sign, e.g. \( x + 5 = 8 \), \( 13x = 39 \), \( 6(x + 3) = 48 \), the student has less need to operate on or with the unknown, or to deal with the equivalence structure of the expressions on both sides of the equal sign. For equations of this arithmetical type the student probably manages to find the value of the unknown by applying known number facts or inverse operations. When the unknown appears on both sides of the equal sign arithmetical understanding is however no longer enough. Neither is arithmetical understanding enough in the abstract type of arithmetical equations where certain algebraic manipulations are needed, for example, because of the presence of negative integers (e.g. \( 2 - x = 7 \)) or several occurrences of the unknown (e.g. \( 6x + 5 - 7x = 27 \)) (see Vlassis, 2002, p. 351).

When solving such more abstract equations the student who has an algebraic understanding of equality first of all acknowledges that the expressions on both sides of the equal sign are representing equal values, next that the solving process involves mathematical actions, which preserve this balance and produce equivalent equations. Vlassis (2002) noted that concrete representations of equalities, like the two pan balance model, might act as good tools for developing students understanding of equality, but Vlassis also pointed at their limitations. For example, the balance model cannot represent the negatives in an equation. More generally, a true algebraic understanding of equation solving implies that the student can consider the equation as representing a concrete problem situation, which means that the student starts to understand the equation as an equivalence structure maintained by the operations one has to apply on both sides to solve for the unknown.

If real-world problem is used as a tool to introduce students to solving equations and the unknown is solved for through the syntax of algebra, the student needs to refrain from an arithmetical understanding of the solution to the problem. This situation may be at odds with the student’s knowledge and intuitions about arithmetic because the meaning of the equal sign changes from announcing a result to stating an equivalence relation (e.g. Carraher, Schliemann, Brizuela & Earnest 2006). Within an algebraic understanding the meaning of the equal sign is relational (Kieran, 1981) and a signal of an equivalence structure. Furthermore, an algebraic interpretation of the solution to the real-word problem implies that the student should be able to refrain from immediately attributing a concrete meaning to the letter appearing in the corresponding equation. Instead the student should understand the letter as an unknown number, the value of which is not significant at the moment the equivalence is set up and manipulated (Vlassis, 2002).

Although the equal sign (=) is taught together with minor than (<) and major than (>) signs in Grade 1 in Finland in order to enhance students’ understanding of equality, students most often use the sign in their mathematical practices in order to show the result of an arithmetic task. It is also usual
to concretize the simple equations during Grade 4 with the balance-scale and encourage students to solve equations with testing, using inverse operations and other informal methods (cf. Cai & Knuth, 2011).

**Methodology**

The material was gathered in the spring 2012 from three Grade 6 classrooms in the Swedish-speaking community of Finland as part of the international VIDEOMAT-study (see Kilhamn & Röj-Lindberg, 2013). Four consecutive lessons on equation solving, and a fifth lesson on problem solving were video-taped and imported into Transana, an open source transcription and analysis software for audio and video data (www.transana.org). The teachers answered a few clarifying questions immediately after each lesson and participated in formal interviews after the last (fifth) videotaped lesson. The teachers Anna, Bror and Cecilia have a similar educational background as certified generalist teachers and Masters of Pedagogy. At the time of the study their teaching experience varied from Bror’s five years to Cecilia’s seven and Anna’s 15 years of experience. They used the same textbook and teacher guide. In this paper we report on a close attention to what happened during the first videotaped lesson when the three teachers introduced solving linear equations in one unknown with their students. We especially considered how the introduction might have supported students’ development of an algebraic understanding of equality. First we briefly present the characteristics of all three teachers’ introductions to solving equations and continue by presenting the case of Anna.

**Three entries to equation solving**

In line with the Grade 6 textbook all three teachers introduced equation solving with one-step equations were the unknown appeared once on the left side and with an integer on the right side. All three took some initiatives in supporting their students’ development of an algebraic understanding of equality but at the same time their use of the simple equations from the textbook restricted them to shed light on the power and need of more formal methods in equation solving.

Cecilia started from a strong emphasis on the equal sign as stating an equivalence relation. Her aim was to help students unlearn their earlier use of the equal sign to represent a string of calculations. She focused on the equal sign as representing equivalence in several ways: by discussing its meaning explicitly with her students, by referring to a solution a student had made in the test, and by representing both inequality and equality with a balance scale. These situations were familiar to the students, who also participated in the corresponding discussions in seemingly relevant ways. However she did not utilize the balance scale analogy to support the emergence of algebraic understanding of equation solving. The first lesson her students solved only equations of the addition type, e.g. \(x + 8 = 15\), by subtracting without any further discussion related to a structural meaning of the equal sign.

Bror started from four uncomplicated real-world situations that were first solved mentally. The first one was “There are seven fruits in a basket altogether. And four of the fruits are apples, the rest are pears, followed by the question: How many pears are there in the basket?” Each situation was then written as an arithmetical equation, e.g \(7 + x = 12\), solved and checked with arithmetical means like
in Cecilia’s classroom. Next Bror did a rapid switch in his teaching to an algebraic interpretation of why the step \( x = 12 - 7 \) in the solution can be thought of as ‘the term 7 is moved to the right side of the equal sign and the corresponding sign is changed’. However, the students’ activity showed no explicit signs of an emerging algebraic understanding of equality. In their verbal answers students continued to refer to the equal sign as a do something –signal, and in all their solutions in the notebooks they used inverse operations to find the value of the unknown.

The third teacher, Anna, started in a similar way as Cecilia and focused on the equal sign as stating an equivalence relation, however in a more formal manner. Anna introduced equation solving by defining what an equation is. She then did a quick transition to explaining the procedure of ‘doing the same thing on both sides’ as a strategy to maintain equality and find the value of an unknown letter in an equation. Her message to the students was clear: you must isolate the unknown number step by step by operating on both sides of the equation.

Here we present two consecutive episodes from the first lesson in Anna’s classroom. The episodes show how language functioned as a tool for memorizing a procedure and supporting the recall of ‘doing the same thing on both sides’ instead of enhancing the students’ understanding of the methods and the need of the use of these methods.

**We have to do the same thing on both sides**

Anna opens her lesson by writing an open number sentence, \( 4 + _ = 9 \), on the white board. As answers to Anna’s questions the students give the value of the open number, and they name the object an equation. Then Anna continues by reading aloud a definition from the white board.

Anna: An equation is an equality relation between two mathematical expressions, which are called left side and right side. It includes one or more unknown numbers. If there is one unknown number, you normally use the letter \( x \).

She then fills the placeholder in \( 4 + _ = 9 \) with the letter \( x \). Next, Anna continues to read aloud: “An equation is an equality relation between two sides. The two sides are separated by an equal sign”. She illustrates the statement both with an arithmetical equality, \( 4 + 2 = 7 - 1 \), where the value of both sides is six, and with the equation \( 4 + x = 9 \) where she emphasizes that both sides of the equation must be equal. The students are asked to solve for two open number sentences and she states that instead of the placeholder they can use a question mark or the letter \( x \). She continues to talk about the convention of using the letters \( x \), \( y \) or \( z \) for an unknown. All the equations she has shown to her students so far include only one number on the right side, except for the arithmetical equality she used to indicate a new understanding of equality: the equal sign as a signal of an equivalence structure.

Before the start of the following episode Anna refers to the procedure of solving equations step by step as a mathematical strategy. She writes the equation \( x + 12 = 18 \) on the white board and starts describing the procedure for solving it.

Anna: An example. X plus twelve is equal to eighteen [She writes on the white board] and we know that x should be six, this is what we know. But also, this way of thinking about how to do it. What we have to aim at is, I want to have (...) If I
have an equal sign in the middle, then I aim at having x alone on the left side (…) But now I have plus twelve there, what do you think, the way of thinking, how can I get this plus twelve away from there? I want to have x alone on the left side of the equal sign. How can I get it away? Janne. Yes, OK.

Janne: Eighteen minus twelve.

Anna: Oh yeah, but now I have it there. What should I do, just to fling it away? How shall I get plus twelve to zero? How shall I get plus twelve to zero? Nelli.

Nelli: Maybe add [inaudible].

Anna: No, if I have, how can I make plus twelve to zero? How shall I get plus twelve to zero, nothing? Olle.

Olle: Maybe change it to x [inaudible].

Anna: No. How shall I get plus twelve to zero? [She draws a minus sign after the number 12 on the left side of the equation]. I have helped you a little bit on the way. Nelli.

Nelli: Minus twelve.

Anna: Minus twelve. But twelve minus twelve is zero, isn’t it. But now the matter here, when I do something on the left side so what do you think I should do on the right side? Tor.

Tor: Take away from there, that twelve.

Anna: Exactly. I have to do the same thing here, now I have got eighteen, what should I also do then, here, on the right side? Well, now, Mimmi.

Mimmi: Minus eighteen.

Anna: No, not minus eighteen, the same thing as on the left side. Mimmi.

Mimmi: Minus twelve.

Anna: Minus twelve. Well let’s check, x, twelve minus twelve is zero, so then, now I’ve got x on the left side, eighteen minus twelve is (…) Quickly Mimmi.

Mimmi: Six.

Anna: Six. Now I have, stepwise, through mathematical steps, done this equation. You could quickly see that it must be six. You could do it just like that. But now I have shown how it actually goes step by step. I want to have x alone on the left side, so that I get what x equals to. And then, I just have to look what I have on that side, what I need to do. In this case, I had plus twelve, then I have to take minus twelve so that it becomes zero. But when I do something on the left side, I also have to do the same thing on the right side. Do you understand? Did you follow?

Immediately after the previous episode, Anna and her students started solving the equation $y - 6 = 11$. In the following episode the step-by-step procedure is repeated and the importance of ‘doing the same on both sides’ is stressed.
Anna: \( y \) minus six equals eleven, an equation. Now, I know that you can; quickly, you know the answer. But, now, we shall think about the mathematical steps. What do I want, I’ll put the equal sign here, what do I want to have alone on this side of the equation? What am I aiming at?

Sofia: \( x \).

Anna: In this case?

Sofia: \( y \)

Anna: \( y \), okay, I’ve got \( y \) there. But it is not ready yet, I’ve got the minus six, what shall I do then, what do I want to do then? Now I’ve got minus six. Cecilia.

Karin: You want to make it zero.

Anna: And how can I get it?

Karin: Plus six.

Anna: Plus six. Okay. And then on the right side I have eleven. Are we ready with it or shall I still do something? What does Janne say?

Janne: Plus six.

Anna: Plus six, too. Why Janne, plus six there too?

Janne: We have to do the same thing on both sides.

Anna: The same thing on the left and right sides. What I do on the left side, the same thing on the right side, or on the right and left sides. It’s plus six, now, because I had minus six. Okay. Then I have got \( y \). Those two cancel each other out. Then I’ve got \( y \) there. And what will be on the right side? Vanja.

Vanja: Seventeen

Anna: Seventeen. And I know that you could have been able, you could find it already in a few seconds, but now we did the mathematical steps, again. Are you following? [SS: Yeah, yeah.] Beginning to understand this, although these are easy numbers (…) This is what you will practice in the book. This is then, now you have solved equations, easy equations. Later, there will be a little bit harder ones, but now we’ll begin with these.

When Anna starts teaching the steps of solving the equation \( x + 12 = 18 \) in the first episode, the students do not contribute with the answers she seems to expect. The answers the students give also show some uncertainty: two students start their answers with “maybe”. After receiving a hint from Anna in the form a minus sign drawn after number twelve in the equation, Nelli gives the expected answer, “minus twelve”. In the second episode we can notice how the students and Anna use the same wordings as when solving the equation \( x + 12 = 18 \) in the first episode. Earlier she stated her expectation very clearly when she said “I want to have \( x \) alone on the left side of the equal sign” and she repeated the question “How shall I get plus twelve to zero?” many times. Now she reformulates
the same expectation as questions to the students “What do I want to have alone on this side of the equation?” followed by “What do I want to do then? Now I’ve got minus six”. And the student Karin contributes with the expected answer “You want to make it zero”. In her summary in the first episode Anna reminded the students “when I do something on the left side, I also have to do the same thing on the right side”. In the second episode the student Janne repeats her words in his answer “We have to do the same thing on both sides” and Anna confirms that he remembers correctly by saying “The same thing on the left and right sides. What I do on the left side, the same thing on the right side, or on the right and left sides”. Anna’s discussion with her students focused strongly on memorizing the procedure and, hence, did not serve well in supporting their development of an algebraic understanding of equality. Solving the equation by algebraic means, and with an algebraic interpretation of equality, was of no use to the students who already knew the value of the unknown x. Nevertheless they tried to fulfill Anna’s expectations and answer her questions.

Discussion

All the three teachers took some initiative in leading their students forward, from an arithmetical to an algebraic understanding of equality. Bror and Anna tried to teach the strategy of doing the same thing to both sides of the equation and Cecilia emphasized the need of understanding the structural meaning of the equal sign. It seems, however, that neither the teachers nor the authors of the Grade 6 textbook were aware of the underlying conceptual differences between solving equations within an arithmetical understanding of equality and, on the other hand, within an algebraic understanding. The students had encountered missing value problems in the textbooks every now and then from the 1st grade onwards. They were familiar with the logic of that type of tasks. However, in the videotaped lessons the students didn’t have any real need to adopt algebraic ways of thinking about equality. For instance, one can wonder whether the students in Bror’s classroom were motivated at all to make sense of the uncomplicated real-world situations with a new complicated way of thinking. The solutions to the problems were obtained more economically by arithmetical means. At best, solving equations by adding or subtracting the same term from both sides of the equation was used by students as procedures among others and applied for one particular type of equation, only. The students did not need an algebraic understanding of the equal sign to solve the tasks, and the book did not explicitly expect students to expand their mathematical knowing into operating with or on the unknown (cf. Filloy & Rojano, 1986).

Balacheff (2001) recommends that students should experience a clear rupture between arithmetic and algebra. The rupture might, for example, be a strong emphasis of the newness of the situation or to give more complex equations to students to be solved. None of the teachers in this study confronted the students with situations where mathematically more powerful approaches were needed than those they were already familiar with. We argue that in many cases algebraic understanding of equality can be developed from just small changes and extensions in the types of tasks presented to students in the elementary school arithmetic, and discussions about them. The students’ arithmetical understanding of the equal sign can be confronted in versatile problem situations where a structural meaning of the equal sign is discussed whilst the situations are
represented and made sense of within an algebraic syntax. Furthermore, by affording the students investigations of series of arithmetic tasks with a pattern, and discussing the pattern, the teacher would help students think generally about numbers and hereby as well support their emerging algebraic understanding of equality (cf. Carraher et al., 2006).

References


