Approaches to learning of linear algebra among engineering students

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The present paper investigates engineering students' own descriptions of what they mean by learning of linear algebra and how they know that they have learned something. I seek to extract keywords from engineering students' descriptions of learning of this discipline by drawing on grounded theory techniques and classifying the answers in conceptual and procedural approaches. By this, both detailed and more meta perspectives on learning are obtained. Results indicate that when explaining their learning of linear algebra, conceptual more than procedural approaches are emphasized. However, in order to know that they have learned something, many engineering students need to know that they are able to solve relevant tasks in the discipline.

Keywords: Approaches to learning, linear algebra, engineering students

Introduction

Students' learning of mathematics is a main interest within the community of researchers in didactics of mathematics. We seek to know how students learn, what they learn, but also how they perceive their own learning (Sfard, 2007). Learning may be defined according to which point of view one has in an investigation, but also by taking into consideration what is relevant for the particular individuals of a study. A classical definition is given by Hiebert and Lefevre (1986), distinguishing between conceptual and procedural knowledge that may yield conceptual and procedural learning. Conceptual knowledge is defined as "knowledge that is rich in relationships" (ibid.1986, p. 6), which means that it cannot exist in isolation. Procedural knowledge includes sequential relationships or step-by-step instructions. Engelbrecht, Bergsten and Kågesten have found conceptual and procedural notions valuable in their research of engineering students (2009), and because the target group of the present investigation is engineering students, these constructs will be utilized.

The present paper focuses on engineering students' interpretation of their own learning in a linear algebra course. Such reflections are beneficial because the students then have to reflect on how they see their mathematical knowledge and for what purposes they study the discipline. Thus, asking questions about learning is valuable and frequently done by researchers. An immediate example is the present data collection, in which questions asked to the students were picked from a research investigation of a related group of students in a mathematics and physics foundation program for students going into an engineering program (Marshall, Summers, & Woolnough, 1999). Based on data from a longitudinal study over an academic year, they derive conceptions of learning held by these students. In my study the setting is somewhat different as the students are experienced engineering students, their reflections about learning are confined to a particular domain in mathematics, and it identifies students' reflections at the end of the course. In this particular setting the following research questions are asked: Which approaches do engineering students include in their description of learning in linear algebra and how do they explain their knowing that they have learned something?

Theoretical Background

The study reported on here investigates engineering students' description of their learning approaches rather than the cognitive processes of learning itself. As will be argued for, such approaches are adequately split in two main categories: approaches connected to conceptual and to procedural knowledge. The definitions were originally given by Hiebert and Lefevre (1986) and are widely used. In this framework, conceptual knowledge is pieces of knowledge connected together or, as explained by Kilpatrick, Swafford, and Findell (2001), "an integrated and functional grasp of mathematical ideas" (p. 118). Procedural knowledge, on the other hand, includes familiarity with symbols but also representation systems in mathematics along with knowledge of rules and procedures that can be used in task solving strategies in mathematics (Hiebert & Lefevre, 1986, p. 6). Some reconceptualizations of these definitions have been put forward, including suggestions about more nuanced terms founded on arguments like superficial and deep quality of the knowledge terms, and extensions and amalgamations of the traditional definitions (e.g.Baroody, Feil, & Johnson, 2007). The interplay between conceptual and procedural knowledge has also been a focus, emphasizing how one knowledge may lead to the other (Rittle-Johnson & Alibali, 1999). Such relationships are multifaceted, and researchers move towards more integrated views in which determining the dynamics between the two is the objective (Engelbrecht et al., 2009).

Students often perceive linear algebra as difficult. This stems from three sources of difficulties (Dorier & Sierpinska, 2001). It is about the pedagogical approach, as proofs are found difficult (Rogalski, 1990). It is also a matter of difficulty with grasping the theoretical concepts and mathematical language; the 'obstacle of formalism' (Dorier, 1997). Finally, linear algebra demands a 'cognitive flexibility' as one has to move between different languages, both theoretical and practical forms. Students tend to think in practical terms (Sierpinska, 2000), and lack of connection to theoretical structures may hinder their learning (Dorier & Sierpinska, 2001).

Engineering students recognize mathematics as a foundation of their education (Khiat, 2010). Still, they consider the discipline as a routine practice of their profession (Steen, 2001) and expect to be exposed to real-world engineering problems in mathematics (Hjalmarson, 2007). This may result in what Kümmerer (2001) denotes a 'workman approach to mathematics' where mathematics is taken as a machine that routinely produces the correct answer if sticking to a set of rules. With such an approach, the formalism of linear algebra may be especially hard to get a grip of. Engelbrecht and colleagues (2009) found that engineering students uphold mathematics as procedurally founded. As part of their investigation, the authors created tailor-made working definitions to focus on engineering students, thus these are adopted in the present study:

"Procedural approach: Use and manipulate mathematical skills, such as calculations, rules, formulae, algorithms and symbols.

Conceptual approach: Show understanding by e.g. interpreting and applying concepts to mathematical situations, translating between verbal, visual (graphical) and formal mathematical expressions and linking relationships." (Engelbrecht et al., 2009, p. 932).

Methodology

The present investigation is part of an ongoing study dealing with engineering students' views about the learning of linear algebra. The teaching format in the course which was taught in English was 'traditional', with large group lectures followed by task solving sessions where students worked in groups. The 'untraditional' part was that a well-functioning video recording system recorded all lectures and published them in-time. The linear algebra course was scheduled in the students' fourth year of studies to become master engineers, postponed in accordance with Carlson's recommendations (1993). However, some basic tools in linear algebra had been introduced in a mathematics course in their first year of studies, since these are necessary for use in the professional disciplines. All together 59 students attended the course this year, and data was collected as I was the teacher and arranged for a questionnaire to be answered at the end of the course. The open questions picked from (Marshall et al., 1999) discussed in the present paper were: "What do you mean by *learning* in linear algebra? And how do you know that you have *learned* something?" Due to experiences from a previous investigation (Rensaa, 2014), the questionnaire was made mandatory but anonymous to increase truthfulness, and the response rate was very good; 93% (55 out of 59).

Data analysis was done in phases. Initially, grounded approaches were used (Strauss & Corbin, 1998) to obtain codes that embrace engineering students' approaches to learning. Next, these codes were related to the conceptual and procedural approaches as described for engineering students (Engelbrecht et al., 2009). Finally, a narrative thematic approach to the statements was performed to deepen their meanings. In the present paper, I concentrate on the quantitative results from coding and organizing in conceptual and procedural approaches. This gives overarching results to answer the research questions about engineering students' approaches to learning.

Analysis and results

The development of codes was done in steps. Initially, I wrote down headwords in each student's description which was given in English. By comparing these, some seemed to describe similar things, e.g., 'utilize for own goals' and 'use in gps' [Global Positioning System], both which could be interpreted as 'learning as applying mathematics'. Because I was working back and forth between statements and codes with an aim of reducing the number of codes without deteriorating their meanings, each time two replies were interpreted within the same category had to be put down as a criterion for the category. For instance, for descriptions of obtained learning, 'know the whole picture' and 'associate theory to applications' were both interpreted as being able to relate the different aspects of linear algebra to each other, thus crystalizing a category called 'ARel' (able to relate). The importance of emphasizing relation in this category was helped forward by a statement that did not fall into this category: 'use different theorems to achieve solutions to practical problems'. The emphasis here is on obtaining solutions more than the relation, thus crystalizing a category called 'ASol' (being able to solve problems). Going back and forth between statements and codes resulted in a final reduction to 8 categories for what learning is and 6 categories for what is meant by learning of linear algebra.

Next, the original data set and my developed codes were sent to another researcher for validation purposes. This researcher used the codes to independently code the data. Then, we met for comparison of results and refinement of codes. A main refinement was deepening the meaning of *applications*. Students had referred to applications when trying to describe learning in linear algebra, but we agreed that students should express that applications were actively studied in a mathematical connection in order to be coded as 'Study Applications' (SAp). An example of a statement where the coding was adjusted by this interpretation is the following:

Student 30: For me, learning is knowing the practical use of theory and how to execute said theory. As a computer engineer student specializing in games development, linear algebra is central in the programming I perform. I only know I have learned something if I can accotiate theory to a problem I encounter.

We agreed that this student is not stating that he is studying applications, but rather that he is actually taking advantage of knowing applications from other disciplines as part of his learning process. Thus, 'Utilize Theory' (UTh) is a closer category as the statement points to how theory may be utilized for practical purposes. The other refinement of codes that was needed was a specification of *relations,* originally named 'Rel'. It was unclear which types of relations this was referring to. The category had derived from students' answers as relating back to previous knowledge, thus the category needed to be adjusted to 'RelB' (relating to background).

Two additional codes were agreed on: the categories 'NoAns' (no answer) and 'Other'. All blank responses could be categorized as 'NoAns', while 'Other' refers to answers that responded to something other than what was asked about. The 'Other' category developed from cases in which divergence in our separate coding appeared. We both encountered problems because none of the codes actually fit with some of the particular answers. An example is 'It really gives the knowledge of different engineering mathematical problems'. One researcher had interpreted this statement as 'Study Applications' (SAp), the other as 'Able to understand why/what is going on' (AUn), but the student does not seem to be actually describing his learning. Thus, the final coding for this response was 'Other'. This joint coding process showed that the codes were adequate and could be used to code all statements. However, we experienced that coding statements together often resulted in finding more information in a reply than what we had done individually.

Ending the process, the following codes crystallized for engineering students' description of what they mean by learning in linear algebra: **SAp** (Study Applications), **GUn** (Gain Understanding), **UTh** (Utilise Theory), **ForM** (Grasp Formalism), **SimP** (Simplify), **SoL** (Solve problems), **RelB** (Relating to Background), and **ToO** (Use Tools). Analytical results for this question are given in Table 1, presenting both the number of students in each category and percentage (rounded off) of the total number of 55 students. The category 'No Answer' consisting of 17 replies is left out, while a number of explanations covered approaches in more than one category. Thus, the sum of percentages does not add up to 100.

	SAp	**GUn**	**UTh**	**ForM**	**SimP**	**SoL**	**RelB**	**ToO**
Number/%	8/15	11/20	10/18	2/4	2/4	11/20	2/4	2/4

Table 1: Responses to what engineering students mean by learning in linear algebra

Coding responses to engineering students' description of how they know that they have learned something gave the following codes: **ASol** (Able to Solve), **AExp** (Able to Explain), **AUn** (Able to Understand Why/What is going on), **AAp** (Able to Apply), **ARel** (Able to Relate), and **ARem** (Able to Remember). Analytical results for this question are given in Table 2, including responses coded as **Other** (answering something else). The table presents both the number of students in each category and percentage and again multiple codes were found in some answers.

	ASol	**AExp**	**AUn**	**AAP**	**ARel**	**ARem**	**Other**
Number/%	15/27	3/5	6/11	9/16	1/2	2/4	5/9

Table 2: Responses to when engineering students know that they have learned something

When the codes and categories were set, I assigned the codes in conceptual and procedural parts. As the codes had developed based on engineering students' own descriptions, they were aligned with Engelbrecht and colleagues' working definition (2009) for conceptual and procedural approaches of engineers. This was done by linking the description of codes to statements given in the definition. Some codes were easier to categorize, like GUn. Gaining understanding was classified as a conceptual approach as this is necessary to be able to expose mathematical understanding. Other classifications were harder. An example is ASol. Problems may be complex, theoretical and demand deep argumentations, and solving these should classify as a conceptual approach. On the other hand, problems may as well be 'standard', connected to a set of skills that are more like a routine part of a learning process. Such dual interpretations of an activity highlight the complexity involved in the distinctions between conceptual and procedural. However, engineering students tend to 'proceduralize' problems, even those of a conceptual nature (Engelbrecht et al., 2009), which is also my experience as teacher. Considering this, I deduced that ASoL ought to be categorized as a procedural approach.

By going back and forth between the definition and codes, a final classification of codes was obtained. For what is meant by learning in linear algebra, the following codes were classified as *conceptual*: SAp fits with 'applying to mathematical situations'; GUn is about 'showing understanding'; UTh may be interpreted as 'translating between verbal and formal mathematical expressions'; and RelB is about 'linking relationships'. The remaining categories were classified as *procedural*: ForM is about 'manipulating' linear algebra expressions; SimP is simplifying by 'calculations'; SoL refers to a way of 'using mathematical skills'; and ToO is to use tools like 'rules, formulas and algorithms'. About knowing that something is learned, the following codes were classified as *conceptual:* AExp is about 'interpreting concepts'; AUn is about 'showing understanding'; AAp is about 'applying concepts to mathematical situations' and ARel is ability to 'link relationships'. The remaining codes were classified as *procedural*: ASol is knowing how to 'use and manipulate mathematical skills'; and ARem may be a part of the manipulation of mathematical skills by recalling how to do this. Drawing on these interpretations, Table 1 and 2 may be organized in conceptual and procedural approaches. Gray coloring of conceptual cells and white coloring of procedural cells indicate the appropriate classification.

Discussion

The analysis results summed up in Table 1 and 2 give a number of indications. In many cases, an interpretation of a student's reply comprised more than one of the codes given in the previous section. An example is the following statement with three codes of a conceptual type and one of a procedural type, codes included in parenthesis:

> Student 6: Generally, I mean that learning is to study something until you understand (GUn) the theory (UTh), and is able to use it in both theoretical and practical problems (SAp and SoL).

In addition, a phrase in a statement could be coded in a mix, as illustrated by the last part of the above statement. Interpreted as being '*able* to use it,' this may be about studying applications as a way of utilizing knowledge in problem solving – SAp, a conceptual approach. If it is interpreted as being 'able to *use* it' this would be more about the solving process itself – SoL; a procedural approach. Thus, a statement could be coded in both procedural and conceptual categories. This is not unique; Engelbrecht and colleagues emphasize that the distinction between conceptual and procedural approaches are complex and not absolute (Engelbrecht et al., 2009). However, being interested in students' own descriptions, the frequency of each code gives a meta perspective on which approaches are most appreciated by engineering students. In this perspective, Table 1 shows that engineering students emphasize conceptual approaches more than procedural ones when explaining what learning in linear algebra means to them.

Table 1 shows that 'Gain Understanding' (GUn) is important to students, having the highest response rate. However, understanding is often – like in the above example – connected to knowing how to *apply* this understanding. Only when being able to apply their knowledge the students think they have understood linear algebra. This result is in line with the fact that these students are engineering students, busy with relating to the use of mathematics (Hjalmarson, 2007). To some students, however, solving of problems becomes the main issue and the scale by which they measure their learning. Lower interest is given to understanding, as the main objective is to obtain a correct answer. An example is the following:

> Student 34: in my opinion, linear equations are some kind of tool (ToO) to solve the problems (SoL) in real industrial areas such as factories and… (AAp).

Not all replies coded as describing learning in a procedural way focus on solving problems. Grasping formalism, which is an aspect of difficulty for students when learning linear algebra (Sierpinska, 2000), may also be interpreted as a procedural approach in terms of manipulating the linear algebra language. This is illustrated in the following student's description:

> Student 5: the meaning of learning linear algebra is actually learning a mathematical language (ForM), a language you can use to solve big questions with many variables (SoL).

Probably connected to students' problems with formalism and expressing themselves in learning terms, there is a notably high number of blank responses; 31% of the students did not give any explanations at all. This may be due to students not bothering to answer the questions, but may also originate from having problems expressing themselves in learning terms. Engineering students, who are not primary interested in mathematics (Kümmerer, 2001) like mathematics students are and who are not focused on learning like teacher students are, may find the questions difficult and avoid them.

Responses to the question about engineering students' knowing that they have learned something, summed up in Table 2, are more equally distributed between procedural and conceptual approaches. This is mainly due to the category 'Able to Solve', which takes all together 27% of the responses. An example of a statement coded within this category is:

Student 35: The simplest way to know that I have learned something is that I can solve some problems (ASol), when I am faced with some practical problems using this method.

This student indirectly says that he seeks to apply the mathematics in practical situations but knowing that he has learned something is concentrated to the solution process itself.

Altogether, a rough answer to the stated research questions may be that the present engineering students emphasize conceptual more than procedural approaches when explaining learning of linear algebra, but in order to know that they have learned something a noteworthy amount need to know that they are able to solve relevant tasks in the discipline. Even if students emphasize understanding it seems like they 'measure' their learning in terms of how ready they are to solve problems. This suggests that emphasis should be put on developing tasks that encourage conceptual approaches. This is not new; developing conceptual problems has for many years been an important aim for teachers, but within linear algebra teaching is often focused around theory and abstractions because they are found difficult by students (Dorier, 1997; Dorier & Sierpinska, 2001). By designing practical related problems in which abstract theoretical arguments are needed, engineering students may familiarize themselves with conceptual approaches to a larger degree. It is still the risk that the students seek to 'proceduralize' such tasks (Engelbrecht et al., 2009), but they offer an opportunity to engage in conceptual arguments on the preferred premises of solving tasks.

Asking students to respond to a questionnaire is not as valued in research terms as asking questions in interviews. It offers the possibility to leave questions open or just respond superficially. Still, the present data material offers valuable feedback that may not be available when doing interviews. One is that *all* students in the class were asked to reply. This gives a higher validity rate of the results, even if not all students answered the questions investigated here. Another benefit is that the response rate to the questionnaire was high, due to the type of method used to collect data. Finally, submitting anonymous answers to a questionnaire may provide more honest answers than when sitting face-to-face with an interviewer. Even if students were asked to reply in writing – which naturally reduces the richness of the replies compared to responding orally – interesting responses were given. The following is an illustration of this, concluding the paper:

Student 9: To learn does not necessarily mean to remember something, but to understand it in depth (GUn) and be able to utilize that information for your own goals (UTh). When one have truly learned something, one can easily explain it to someone else (AExp).

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