Mathematical memory revisited: mathematical problem solving by high achieving students

Attila Szabo

Stockholm University, Department of Mathematics and Science Education, Sweden; attila.szabo@stockholm.se

The present study deals with the role of the mathematical memory in problem solving. To examine that, two problem-solving activities of high achieving pupils from secondary school were observed one year apart - the proposed tasks were new and challenging for the pupils, but could be solved with similar methods. The study shows that even though not recalling the previously solved task, the pupils' individual ways of approaching both tasks were identical. The study also displays that the participants used their mathematical memory mostly at the initial phase and during a small fragment of their problem-solving process, and it indicates that those pupils who apply algebraic methods are more successful than pupils who use numerical approaches.

Keywords: high-achievers, mathematical memory, mathematical abilities, problem solving

Introduction and background

Despite a growing emphasis on the identification and teaching of mathematically able pupils, much remains unknown about the abilities they display when solving mathematical problems. For reasons of social justice and equality, research has so far focused on low achieving pupils (Swanson & Jerman, 2006) and only a restricted amount of studies have observed the abilities of the gifted and high-achievers (e.g. Vilkomir & O´Donoghue, 2009) or focused those pupils memory functions during mathematical activities (Leikin, Paz-Baruch, & Leikin, 2013; Raghubar, Barnes, & Hecht, 2010). Besides, just a few studies (e.g. Krutetskii, 1976; Szabo, 2015) examined the role of the *mathematical memory* in gifted and talented pupils' problem-solving activities.

Mathematical abilities

Our innate ability to estimate quantities, known as the approximate number system, is extremely limited (Dehaene, 1997), but an active contact with the subject may, under favourable conditions, generate mathematical abilities that are both complex and structured (Krutetskii, 1976). The nature of mathematical abilities has engaged researchers for more than 120 years; already at the end of the 19th century, Calkins (1894) presented, based on observations of Harvard students, significant information about the way mathematicians approached the subject. However, the research on mathematical abilities - mainly because of the dominance of psychometric approaches, and thereby considering abilities as innate and static - has not delivered widely accepted results during the first half of the 20th century (Vilkomir & O´Donoghue, 2009). Therefore, of substance for the present paper is the research of Krutetskii (1976), who analysed the problem-solving activities of around 200 pupils in a longitudinal observational study; he conclude that able pupils' mathematical ability is both complex and dynamic, with the following abilities: a) *obtaining and formalizing mathematical information* (e.g. formalized perception of mathematic material), b) *processing* *mathematical information* (e.g. logical thought, flexibility in mental processes, striving for clarity and simplicity of solutions, generalizing mathematical relations and operations), c) *retaining mathematical information*, that is, *mathematical memory* (a generalized memory for mathematical relationships) and d) a *general synthetic component*, named "mathematical cast of mind" (Krutetskii, 1976, pp. 350–351). Even though studies (e.g. Szabo, 2015) demonstrate that high-achievers are not necessarily mathematically gifted - and even if Krutetskii's model is frequently used to identify mathematical giftedness - Krutetskii (1976, pp. 67–70) underline that high-achievers are also able to manifest accurate mathematical abilities.

Mathematical memory

It is largely agreed that memory plays an essential role both in the learning of mathematics and in mathematical problem solving (e.g. Leikin et al., 2013; Raghubar et al., 2010). Thus, what it seems to be crucial "is not whether memory plays a role in understanding mathematics but what it is that is remembered and how it is remembered by those who understand it" (Byers & Erlwanger, 1985, p. 261). Calkins (1894) early study displayed that the memories of mathematicians are rather concrete than verbal, that mathematicians do not memorise facts easier than other students, and that, when performing mathematics, there is no difference between men and women. Even though studies conducted during the following decades addressed memory functions in quantitative terms, Katona (1940) indicated that rational methods are easier to memorise that random digits and Bruner (1962) showed that simple interrelated representations are effective when recalling detailed knowledge. Krutetskii (1976) differentiates *mathematical memory* from the *mechanical recalling* of numbers or algorithms, by stressing that it is a memory consisting of generalized methods for problem solving. Hence, the mathematical memory is not retaining "all of the mathematical information that enters it, but primarily that which is 'refined' of concrete data and which represents generalized and curtailed structures" (Krutetskii, 1976, p. 300). And it was also observed that able pupils usually retain the contextual facts of a problem only during problem-solving and forgot it instantly afterwards, but they remember several months later the methods they applied when solving the problem. Conversely, low-achievers often remember contextual facts, but hardly ever the problem-solving methods (Krutetskii, 1976). Cognitive psychology studies (e.g. Sternberg & Sternberg, 2012) indicate important distinctions between different memory systems; that is, long term memory can be divided into *implicit* and *explicit* memory, based on the type of the stored information. In a mathematical context, the implicit memory contains automatized procedures and algorithms, while explicit memory retains information about experiences and facts which can be consciously recalled and explained, such as schemas for problem-solving. Thus, according to the cognitive model, we may assume that mathematical memory, as defined by Krutetskii, is explicit. Besides, it is a memory formed at later stages (e.g. Davis, Hill, & Smith, 2000) based on the ability to generalize mathematical material, because at young able pupils "the relevant and the irrelevant, the necessary and the unnecessary are retained side by side in their memories" (Krutetskii, 1976, p. 339).

The Study

The present study had two aims, both drawn on Krutetskii's (1976) definitions of mathematical ability. First, to display the structure of the mathematical ability when high-achieving pupils solve problems which are new and challenging, but can be solved with similar methods. Secondly, to examine the role of the mathematical memory during the mentioned problem-solving activities.

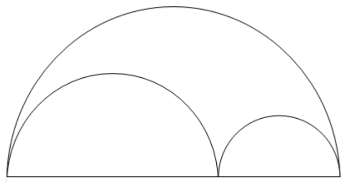
Participants

As indicated, young children and low-achievers typically don't manifest proper forms of mathematical memory; conversely, older high-achieving pupils are able to display higher order mathematical abilities (Krutetskii, 1976). Thus, the participants were 16-17 years old, from an advanced mathematics programme in Swedish secondary school and achieved the highest grade in mathematics. The participation was optional. Prior to observations, to familiarize pupils with the study, I spent 30 hours, over a period of four months, as a participant observer in their mathematics classroom. During this period they started to treat me as a mathematical peer and, importantly, they trusted me as an observer of their problem-solving activities. At the end of this process, after consulting their teacher, 6 students, 3 boys and 3 girls, were selected to participate in the study.

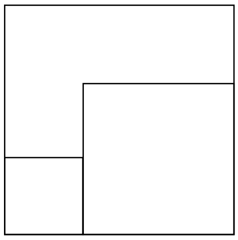
Tasks

The theoretical background indicates that an appropriate way to identify the distinct structure of the mathematical ability is to analyse the problem-solving activities of the individuals (e.g. Krutetskii, 1976). Moreover, the structure of a mathematical problem reveals the mathematical thinking which is required to solve it, because problem solving "is an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine" (Cai & Lester, 2005, p. 221). However, able pupils typically forget the context of a problem shortly after solving it, but, as an impact of their mathematical memory, they are several months later able to recall the methods applied to solve it. Thus, in order to complete the aims of the study, the participants solved two problems approximately one year apart. At the first observation, in order to avoid as far as possible the influence of previous experiences, the main criterion was to select a new and challenging task, Task 1 (T1). When selecting Task 2 (T2) - in order to emphasize the mathematical memory - the main criterion was to propose a task which was new and challenging, but could be solved with methods similar to those used previously.

Task 1: In a semicircle we draw two additional semicircles, according to the figure. Is the length of the large semicircle longer, shorter or equal to the sum of the lengths of the two smaller semicircles? Justify your answer.



Task 2: In a square we draw two arbitrary contiguous squares, according to the figure. Is the perimeter of the large square longer, shorter or equal to the sum of the perimeters of the two smaller squares? Answer the question without measuring the figure. Justify your answer.



Both tasks underwent a substantial a-priori testing with corresponding groups of high-achievers, confirming that they were well-suited for the study and for the mathematical knowledge of the participants. This test confirmed that the pupils solved the proposed tasks with similar methods, that is, by applying the formulae for perimeters of circles and squares.

Observations and interviews

To avoid confounding factors during classroom interaction, which affect pupils' thought process (Norris, 2002), every pupil was observed individually; to avoid stress, they had unlimited time to solve the tasks. The observations occurred at two distinct occasions: first, T1 was solved and approximately one year later T2 was solved. In order to avoid that the participants' memories will be activated mostly because of recalling the circumstances for the first observation as an unusual element in their daily activities - that is, not because of recalling the previously solved task - I continued to interact with them at their mathematics classes during the time period between the two observations. To minimise the pupils' influence on each other, the tasks were solved during single days. The pupils were invited to solve the tasks in a think-aloud manner and encouraged to describe every step in the process. The observations took place in a private room at their school and, when needed, supplementary questions were posed in order to facilitate the process. If a pupil neither wrote nor spoke for a while, the following questions were posed: What is bothering you? Why do you do that? What do you want to do and why? What are you thinking about? Pupils generally are not used to verbalise their problem-solving process (Ginsburg, 1981), thus, in order to avoid the risk that essential parts of their cognitive activities would not be communicated, every observation was followed by a *reflective interview*. The purpose of the interviews was to display the hidden cognitive processes at problem-solving and to evaluate the levels of competence in those processes (Ginsburg, 1981). Each observation was recorded using a technology which enables to digitalise both speech and handwritten notes; the recordings were transcribed verbatim. Although they were given unlimited time, no participant needed more than 14 minutes to solve a single task.

Data analysis

The piloting of the tasks on corresponding groups of high-achievers indicated that the *general synthetic component* - a typical ability of gifted pupils (Krutetskii, 1976, p. 351) - was not observable during problem-solving; consequently, this ability was excluded from the analysis. The *ability to generalize* mathematical relations is a prerequisite of mathematical memory (Krutetskii, 1976, p. 341), thus, its occurrence was also investigated. The analytical framework for this study contained the following abilities from Krutetskii's framework: *obtaining* and *formalizing* mathematical information (O), *processing* mathematical information (P), *generalizing* mathematical relations and operations (G), and *mathematical memory* (M).

The digital recordings resulted in an exact linear reproduction of the pupils' actions, which was especially beneficial when performing *qualitative content analysis* of the material, inspired by van Leeuwen (2005). The pupils´ actions were analysed by identifying, coding and categorising the basic patterns in the empirical content. This method highlighted the abilities that were directly expressed in the empirical material; each episode lasting at least one second in written solutions and verbal utterances was scrutinised for the presence of the focused abilities. Next, the data from observations were combined with data from the interviews. I exemplify this with data from Linda, who, when solving T2, didn't say or wrote anything during the initial 62 seconds, before stating:

Linda: I would like to write down, start with writing a… some nice little estimations…

After this episode she drew three squares with sides *a*, *b* and *c*, and wrote "$a + b = c$". Thus, based on the observation, the presence of O was certain, but it was not possible to decide if other abilities were also present in the actual episode. Yet, the following sentences from the reflective interview proved that she recalled another task which could be solved with similar methods:

Linda: I got blocked until I remember similar tasks, because it's a lot more difficult to solve this kind of tasks if one doesn't have a determined way to approach it… I believe I will bring up the same task as last time, with triangles and squares.

The utterances "a determined way to approach it" and "the same task as last time" indicate that Linda recalled a different task and its methods, thereby validating the presence of both O and M in the actual episode. In this way, the analysis revealed both the structure and the sequential order of the focused abilities, that is, every ability which occurred during the 12 problem-solving activities was displayed in a matrix. However, as exemplified above, some abilities (e.g. O and M) occurred closely interrelated during certain episodes and were extremely hard to differentiate.

Results

When asked, each pupil confirmed that both tasks were new and challenging, this being a prerequisite for the study. The analysis concluded in a matrix, with every episode of the process related to the focused abilities. As mentioned, certain abilities were closely interrelated during some episodes. As displayed (Table 1), M is present - solitary or interrelated - at 16% during the first and at 10.5% during the second observation. The most manifested ability is P, which increased from 53% to 67% a year later, while O, the second most exposed ability, decreased from 47% to 31.5%.

	O	**O & P**	**O & M**	**P**	**G**	**M**
Task 1	31%	4%	12%	49%	0%	4%
Task 2	20%	1.5%	10%	65.5%	2%	0.5%

Table 1: Average time for the focused mathematical abilities in the problem-solving process

According to the a-priori testing of the tasks, G could be revealed when numerical results - that is, solutions for particular cases - were developed into general, algebraic solutions. Thus, every pupil who presented purely numerical results, that is, Erin, Sebastian and Larry, was encouraged to develop general solutions. Yet, when solving T1 and asked if their numerical results apply also for arbitrary semicircles, none of them could generalize (G) their findings:

Erin: I don't know how I should prove this … if I have to do some general method.

Sebastian: I don't know if I shall demonstrate that it should be the same thing there, for every measure. But now in my head it sounds like that it should be so.

Larry: Yes, I suppose, but I don't know how to confirm it, it only feels that way.

Thus, the increase of G from 0% to 2% (Table 1) occurred because during the reflective interview after T2 Erin performed a successful generalization of her numerical results, and stated:

Erin: I've never made a general solution like this ... But it was fun ... Especially when it concluded in something.

When concerning the efficiency of the applied methods, the analysis shows that Earl, Linda and Heather solved both tasks properly by applying general, algebraic methods. Conversely, purely numerical approaches didn't lead to fully acceptable results. The most efficient solutions were offered by Linda, who applied the same algebraic model (and its identical steps) at both tasks.

The role of mathematical memory

The recalling of the applied methods several months after solving a problem is a typical display of mathematical memory (Krutetskii, 1976). Thus, another main criterion for the study was that both tasks could be solved with similar methods. However, only Earl and Larry associated T2 to T1:

Earl: We got a very similar task last year, when we had the circle and that semicircle.

Larry: We did a pretty similar task last time, when it was something like this, something with the radius or diameter on them.

Earl and Larry applied identical methods at the individual level when approaching both tasks. That is, Earl solved both tasks by using the same algebraic method, while Larry approached both tasks with the same numerical method. However, Earl's algebraic method gave accurate solutions while Larry couldn't solve any task properly. The other four pupils said that they didn't associate T2 to T1. But even though not recalling T1, they approached both tasks in identical ways at the individual level. For example, when Linda solved T2, despite stating that she didn't think at all of T1, she applied the same general method as a year before:

Linda: I will bring up the same task as last time, with triangles and squares. It is a bit the same thing ... I connect very often geometrical tasks to that. I have written that solution many times and I can see every step in the process in front of me.

As seen above, Linda refers to a generalized method which she associates to a geometrical task - about finding the side of a square drawn in a right triangle - which differs considerably from the proposed tasks. Yet, influenced by her mathematical memory (Krutetskii, 1976, p. 300) she states that "It is a bit the same thing" and applies the same method when solving both T1 and T2.

Heather as well used identical algebraic approaches for both tasks a year apart:

Heather (T1): I needed a common variable. Otherwise it will be difficult to calculate.

Heather (T2): I needed some relation among these sides in that and the large square's sides. Otherwise it will be difficult.

Also the individual approaches of Erin and Sebastian were respectively identical; Erin approached both tasks by *reasoning*, testing *numerical values* and applying *particular solutions*, while Sebastian *reasoned* carefully before requesting the use of *numerical values* at both occasions. Thus, every pupil approached both tasks identically at the individual level. The analysis also shows that M is displayed mainly at the beginning of the process, for recalling mathematical relations and problem-solving methods; moreover, none of the participants modified their initially selected methods.

Discussion

One of the aims of this study was to display the role of the mathematical memory (M) when high-achieving pupils solve new and challenging tasks, which can be solved with similar methods. Despite its small proportion, M seems to play a pivotal role in problem-solving because the pupils selected their methods at the early stages of the process and the methods were not changed later. Thus, by confirming earlier results (e.g. Szabo, 2015), it seems that the choice of methods is directly influenced by M and it is critical for the success of the problem-solving.

However, only two of six participants recalling the previously solved task is somehow unexpected, according to Krutetskii's (1976) description of M. But even when not recalling T1, every pupil approached both tasks in the same individual way. For example, Linda's method (connected to a square in a triangle) is apparently very different from what is expected, yet it is a general approach that she uses for every novel geometrical task. But even though the individual approaches of Erin, Larry and Sebastian were not successful when solving T1, they were reproduced exactly a year later. Thus, it seems that the participants rely on methods which are inflexible according to their details and are applied regardless if they are successful or not. The general structure of the participants' mathematical abilities shows that O and M decrease while P increases at the second observation. Thus, it is not unreasonable to assume that the displayed stability of the individual approaches made O more efficient at T2, and consequently pupils had a larger focus on P. And even though none of the pupils could generalize numerical results at the first observation, Erin generalized her results during the interview after T2. Thus, when additional opportunities were offered, Erin could improve the quality of her problem solving. This finding may suggests that some of the participants have unlikely experienced teaching focused on methods of generalization, because the individual structure of the mathematical ability depends on the received instructions (Krutetskii, 1976).

By confirming earlier studies (e.g. Krutetskii, 1976), the methods of this study were not sufficient to differentiate M from O at those episodes when pupils didn't say nor write anything. Thus, a better investigation of the mathematical memory requires further studies, possibly with approaches from the field of cognitive neuroscience. Finally, the present study highlights some important issues. First, that individual problem-solving methods are very stable and apparently independent of their efficiency when high-achievers solve new, challenging tasks. Secondly, that mathematical memory has a key role during the early stages of problem-solving. Thirdly, that if they aren't familiar with algebraic methods, high-achievers might fail to display higher order abilities associated with giftedness, such as the ability to generalize mathematical relations and operations.

References

Bruner, J. S. (1962). *The process of education.* Cambridge: Harvard University Press.

Byers, V., & Erlwanger, S. (1985). Content and form in mathematics. *Educational Studies in Mathematics*, *15*(3), 259–275.

Cai, J., & Lester, F. A. (2005). Solution representations and pedagogical representations in Chinese and U.S. mathematics classrooms. *Journal of Mathematical Behavior, 24*(3), 221–237.

Calkins, M. W. (1894). A study of the mathematical consciousness. *Educational Review, VIII*, 269–286.

Davis, G., Hill, D., & Smith, N. (2000). A memory-based model for aspects of mathematics teaching. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, 2* (pp. 225–232). Hiroshima: Hiroshima University.

Dehaene, S. (1997). *The number sense*. New York: Oxford University Press.

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: aims, rationales, techniques. *For the Learning of Mathematics, 1*(3), 4–11.

Katona, G. (1940). *Organizing and memorizing: Studies in the psychology of learning and teaching.* New York: Columbia University Press.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: The University of Chicago Press.

Leikin, M., Paz-Baruch, N., & Leikin, R. (2013). Memory abilities in generally gifted and excelling-in-mathematics adolescents. *Intelligence, 41*(5), 566–578.

Norris, S. (2002). The implication of visual research for discourse analysis: transcription. *Visual Communication, 1*(1), 97–121.

Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences, 20*(2), 110–122.

Sternberg, R.J., & Sternberg K. (2012). *Cognitive Psychology*. Belmont, CA: Cengage Learning.

Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research, 76*(2), 249–274.

Szabo, A. (2015). Mathematical problem-solving by high achieving students: Interaction of mathematical abilities and the role of the mathematical memory. In K. Krainer & N. Vondrová (Eds.), *Proceedings of CERME9* (pp. 1087–1093). Prague: Charles University, ERME.

van Leeuwen, T. (2005). *Introducing social semiotics.* London: Routhledge.

Vilkomir, T., & O´Donoghue, J. (2009). Using components of mathematical ability for initial development and identification of mathematically promising students. *International Journal of Mathematical Education in Science and Technology, 40*(2), 183–199.