# Promoting prospective elementary teachers' knowledge about the role of assumptions in mathematical activity

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*Although the notion of assumptions is important in the discipline of mathematics and permeates (often tacitly) mathematical activity in school mathematics, instruction in the elementary school pays little attention to it. This situation is unlikely to change unless elementary teachers have a good understanding of the role of assumptions in mathematical activity and an appreciation of the pedagogical implications of that role. In this paper, we investigate how teacher education can promote this dual goal by illustrating a promising, task-based approach to support prospective elementary teachers to develop pedagogically functional mathematical knowledge about the role of assumptions in mathematical activity. We developed the approach in a 4-year design experiment we conducted in a mathematics course for prospective elementary teachers in the United States.*

*Keywords: Assumptions, elementary grades, prospective teachers, task design, teacher knowledge.*

## Introduction

Assumptions denote the statements doers of mathematics use or accept (often implicitly) and on which their claims are based (Fawcett, 1938) and, thus, assumptions are fundamental to any mathematical activity at all levels. In school mathematics, however, assumptions often receive little explicit attention, especially in the elementary school. The notion of assumptions has received also relatively little explicit attention in the mathematics education literature.

A small number of studies addressed the role of assumptions in the particular area of proving at the secondary (Fawcett, 1938; Jahnke & Wambach, 2013) and elementary (Stylianides, 2016) school levels. However, there is a scarcity of relevant research at the teacher education level. This is problematic: unless teachers are supported to develop good understanding of the role of assumptions in mathematical activity and an appreciation of its pedagogical implications, it is unlikely that teachers will offer to their students productive learning opportunities regarding assumptions.

In this paper, we investigate how teacher education can promote prospective elementary teachers' knowledge about the role of assumptions in mathematical activity. We focus on the design and implementation of tasks that can support *Mathematical Knowledge for Teaching* (*MKfT*).

## Research background

The notion of MKfT (Ball et al., 2008) denotes the kind of "pedagogically functional mathematical knowledge" (Ball & Bass, 2000, p. 95) that teachers need to have to be able to manage the mathematical demands of their practice. It has been noted that "there is a specificity to the mathematics that teachers need to know and know how to use" (Adler & Davis, 2006, p. 271) and that teacher education should aim to create learning opportunities for prospective teachers (PTs)

that would enable them "not only to know, but to learn to use what they know in the varied contexts of [their] practice (Ball & Bass, 2000, p. 95). Thus mathematics courses for PTs cannot lose sight of the domain of application of the targeted knowledge (i.e., the domain of mathematics teaching).

We consider next what might be essential elements of MKfT about the role of assumptions in mathematical activity. We begin by describing two elements we identified based on consideration of the role of assumptions in the discipline of mathematics (e.g., Fawcett, 1938; Kitcher, 1984) and mathematical analysis of classroom practice where the notion of assumptions received explicit instructional treatment (Fawcett, 1938; Jahnke & Wambach, 2013; Stylianides, 2016).

- *Element 1:* Understanding that a conclusion is dependent on the assumptions on which the argument that led to it was based.
- *Element 2:* Understanding that different legitimate assumptions can lead to different conclusions that, although on the surface may appear to be contradictory to each other, may be valid within the set of their underpinning assumptions.

These interrelated elements are central to mathematical work. The different sets of axioms in Euclidean and non-Euclidean geometries, including associated results, offer an illustration of both elements in the discipline of mathematics or the upper secondary school (Fawcett, 1938). Yet similar ideas are central also to the elementary school (Stylianides, 2016) and are consistent with recommendations in curriculum frameworks (e.g., NGA & CCSSO, 2010, p. 6). Furthermore, the two elements are essential for mathematical knowledge that is also pedagogically functional. For example, they can allow a teacher to recognize that two students who offered inconsistent answers to a task might have applied sound reasoning based on different assumptions. We capture this kind of examples of pedagogical functionality of elements 1 and 2 under a third element of MKfT:

- *Element 3:* Ability to recognize possible ways in which elements 1 and 2 might apply in mathematics teaching.

Regarding how to promote PTs' MKfT, in Stylianides and Stylianides (2014) we discussed and illustrated a special kind of *mathematics* tasks that we called *pedagogy-related mathematics tasks*. These tasks have two major features: (1) A mathematical focus, which relates to mathematical ideas that are important for teachers to know; and (2) A substantial pedagogical context, which is an integral part of the task and essential for its solution. Notwithstanding the importance of pedagogy-related mathematics tasks, there is also a need for the use of other kinds of tasks in mathematics courses for teachers. In this paper, we focus on a task sequence that illustrates another approach we followed to promote MKfT. Unlike pedagogy-related mathematics tasks where mathematics and pedagogy are intertwined in the same task, this task sequence illustrates an approach to promoting MKfT whereby initial work on a mathematics task can be not only mathematically but also pedagogically generative, with follow-up instruction fostering the intertwinement between mathematics and pedagogy. We elaborate on the task sequence in the Method section.

To conclude, in this paper we address the following research question: What task sequence can offer a productive learning environment for prospective elementary teachers to develop the three elements of MKfT that we described earlier about the role of assumptions in mathematical activity?

## Method

### Research context

This research derived from the last cycle of a 4-year *design experiment* (e.g., Cobb et al., 2003) in a semester-long mathematics course (3hrs per week) for prospective elementary teachers in the United States. The design experiment comprised five cycles of implementation, analysis, and refinement of task sequences and associated implementation plans that aimed to promote PTs' MKfT. The students in the course were undergraduates who majored in different fields and were taking the course as a prerequisite for admission to the master's level elementary teacher education program. The task sequence we focus on was implemented toward the end of the semester and it was the only one that explicitly aimed to promote PTs' MKfT about the role of assumptions in mathematical activity. We introduced the task sequence in cycle 4 because we felt an explicit intervention was needed to adequately promote PTs' MKfT in relation to elements 1–3. Analysis of how the task sequence played out in cycle 4, alongside our developing understanding of how things "worked," led to modifications of the task sequence, culminating in the form described below.
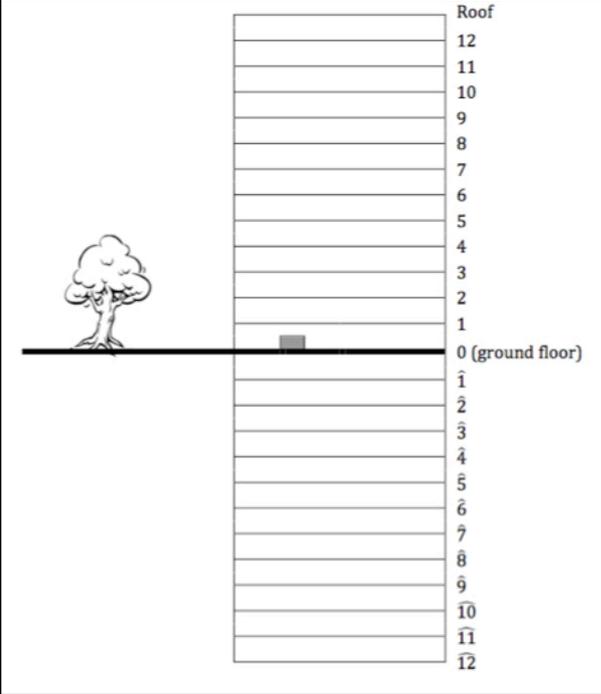


**Figure 1: The "Floors Problem" (derived from Ball, 1993) in part A of the task sequence**

### The task sequence

The task sequence comprised three parts. In part A we used the "Floors Problem" (Ball, 1993) in Figure 1. Two task features made it suitable for our goals to promote elements 1 and 2 of MKfT. First, the task conditions were ambiguous and thus subject to different legitimate assumptions; a major ambiguity concerned whether or not the person in the task had to travel directly to the second floor. Second, different assumptions about the task conditions could support different arguments for answers to the task that might appear to be contradictory but might actually be correct given their

underpinning assumptions. In addition to promoting elements 1 and 2 of MKfT, we hypothesized that PTs' mathematical experience with the task in part A could offer a productive context for pedagogical reflection (cf. element 3). We capitalized on this potential of the task in parts B and C.

In part B we used two *conceptual awareness pillars* (or simply *pillars*; Stylianides & Stylianides, 2009) to which the PTs responded individually and in writing. (There was a third pillar whose discussion we omit due to space limitations and a lack of direct relevance to our focus.) The pillars are presented in the first column of Table 1 (see next section). With them we aimed to amplify PTs' mathematical learning (see pillar 1 in relation to elements 1 and 2 of MKfT) and its intertwinement with pedagogical reflection (see pillar 2 in relation to element 3). As per the definition of pillars (Stylianides & Stylianides, 2009), the non-directive prompts in them could increase PTs' awareness of key realizations they might have developed (possibly in tacit form) during their work on the task.

Finally, in part C we engaged PTs in small-group and whole-class discussion around the three prompts in Figure 2 that aimed to further support PTs' reflective insights from part B and help them consider the applicability of those insights beyond the particular task. The prompts raised issues relating to elements 1–3 of MKfT. For example, in prompt 1 the first question related to elements 1 and 2, while the second question related to element 3. Each small group was asked to collectively produce a written response to each prompt prior to a whole-class discussion. Due to space constraints we report only findings from our analysis of the small groups' written responses.

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1. "Conclusions are 'true' only within the limits of the assumptions on which they are based." How do you understand this statement? Do you think it is important for elementary school students to develop a sense of the role of assumptions in mathematics? Explain.

2. "Teachers should always make sure that the mathematical tasks they give to their students have unambiguous conditions." What do you think about this statement? Explain.

3. There may be situations where teachers do not realize that a mathematical task they give to their students has ambiguous conditions. What might happen in these situations and how might teachers handle the situations?

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**Figure 2: Discussion prompts used in part C of the task sequence**

**Data and analysis**

The task sequence in research cycle 5 was implemented in two parallel (independent) classes of the course that were both taught by the second author. At this stage of our analysis we are using data from only one of the classes: videos and transcripts of the implementation of the task sequence in this class with 16 PTs attending on that day; these PTs' written work including their individual responses to part B and their group responses to part C; and field notes from a research assistant documenting the work of one small group. Our data analysis was guided by elements 1–3 of MKfT that we aimed to promote. Specifically, we used qualitative content analysis to analyze the transcripts and field notes in part A to examine whether and how the PTs developed understandings relating to elements 1 and 2, and also PTs' responses to parts B and C to identify themes in their responses and examine whether and how these themes corresponded to elements 1–3 of MKfT.

## Implementation and discussion of the task sequence

### Part A: implementation of the Floors Problem

Stylianides showed the problem statement and the building model, and explained the notation for negative numbers. He also established a common notation with the class about representing trips. After that, he asked the PTs to work on the problem first individually and then in groups of four. He also said he would not answer any clarifying questions about the problem, an intentional aspect of our instructional design. The discussion in the small group where the research assistant was taking field notes is indicative of the way the PTs engaged with the problem (all names are pseudonyms):

> Amanda: So… can you pass the second floor and then go back to it? Or do you have to stop, because you've technically gotten there? So… you just have to look at how to get *directly* there? If not, it's going to be infinity!
>
> Beth: That's what I did…the direct. I just counted how many up and how many down…
>
> Monica: But that doesn't say that you have to come in the entrance… So you can start anywhere?

At this point Stylianides stopped by the small group and Monica asked him whether the trips had to start on the ground floor. Stylianides reminded her that he would rather not respond to clarifying questions and he moved on to a different group. The small group discussion continued as follows:

> Victor: So… the solution of this problem depends on our assumption.
>
> Amanda: Well, fine then. I say 15… because I'm assuming that you start on the ground floor and can't pass [floor] 2 and can't change direction more than once.
>
> Victor: I say 25 because… [He was interrupted by the start of the whole-class discussion.]

As illustrated by this exchange, the decision not to respond to clarifying questions allowed for the notion of assumptions to emerge naturally in the discussion: had the instructor specified an interpretation, the PTs would not have considered alternative interpretations. The whole class discussion started with the small groups explaining their answers to the problem: 25, infinity, 15, and 51. Sherrill explained the answer of 25 by noting how each floor could constitute a single starting point for a direct trip to the second floor. Sophie then tried to explain infinity as an answer:

> Sophie: Well, we picked infinity because […] if you're saying you can go up and down and up and down as many times as you wanted before you reach the second floor, I think that the question isn't specific enough. Like they [Sherrill's group] assumed you can only go up or down one time. […]
>
> Stylianides: So you say [referring to Sophie] if we don't assume that [what Sherrill's group assumed] and [we assume] you're allowed to travel up and down as many times as you want, then the answer would be infinity? [To the class:] What do you think about that? Is one of them wrong? Are both of them right? […]
>
> Lindsey: I think […] they just made different rules, and like, thought processes. […] You could have a direct route and that's it, no going back and forth… like, you *could*,

when you were talking about going up and down and up and down and that's not really a direct route… So really, you could do anything that you wanted to, but it just depends on what the person… what the problem is looking for, I guess.

Sherrill:     It's up to the interpretation of the reader... Like I read it and I just assumed that it had to be a direct route, but it clearly doesn't state that… […]

The answer of infinity was discussed again later, after the class had considered also the answers of 15 and 51. Overall, the whole-class discussion mirrored the previous small-group discussion. Sophie, Lindsey, and Sherrill's comments show an increased understanding of elements 1 and 2: the PTs acknowledged that the task allowed for different legitimate interpretations of its conditions and that their conclusions depended on their assumptions (element 1), while realizing that the different answers they came up with were all correct based on their underpinning assumptions (element 2).

**Part B: PTs' responses to the pillars**

Analysis of PTs' responses to the two pillars gave rise to the themes summarized in Table 1. Each response could receive multiple codes. The frequencies are offered to show how prominent each theme was among PTs' responses, not for generalizability. Regarding pillar 1, PTs' responses under themes 1–4 related to both elements 1 and 2 of MKfT. The responses below illustrate all themes:

Sherrill: I was surprised at the multiple assumptions made. I had not even considered the fact that answers other than 25 existed because of the way people interpreted the problem. I now realize how greatly making assumptions can alter mathematics. (themes 2, 4)

Lorri: [A]ll of our groups had different interpretations of how many ways there were […] Some of the other groups' answers […] I could understand their rationales as to why their answer was correct. Our answers were not wrong under this ambiguous problem. (themes 1, 3)

| Pillars | Themes and frequencies (in parentheses) |
|---|---|
| *Pillar 1*: Is there anything that particularly stood out to you from our work on the "Floors Problem"? | 1.  Multiple legitimate interpretations (8)<br>2.  Multiple assumptions (5)<br>3.  Multiple correct answers (4)<br>4.  Importance of assumptions (4)<br>5.  Other (2) |
| *Pillar 2*: The "Floors Problem" was purposefully designed to be ambiguous. Why might a teacher use a problem like that in his/her classroom? | a.  To encourage sensitivity to language (7)<br>b.  To increase awareness of the role of assumptions (6)<br>c.  To enhance appreciation of different possible interpretations of the same (ambiguous) text (6)<br>d.  To enhance appreciation of the interdependency between different assumptions/interpretations and different answers and solution paths (5)<br>e. To encourage discussion, explanation, or proof (5)<br>f.  Other (3) |

**Table 1: Summary of results from PTs' responses to the pillars in part B of the task sequence (*n*=16)**

Regarding pillar 2, PTs' responses under themes a–e showed ability to recognize how their mathematical insights from the problem could apply in teaching. Thus the responses offered evidence related to element 3 of MKfT. We offer two responses that illustrate some of the themes:

> Lorri: A teacher could use this ambiguous problem to get their students to understand that sometimes their answers are not wrong and that there may be other answers to a problem. Students will be able to explore their different ways of interpretation and must be open to others' interpretations […] They will understand that there will be different […] answers that will depend on the information given in the problem, which may be ambiguous. (themes a, c, d)

> Helen: To learn about assumptions and understand not only the importance of specifics […] but also the freedom for one to think along the terms of their own assumptions. (themes a, b, c)

**Part C: responses of small groups to the discussion prompts**

Overall the PTs' responses to part C were pedagogically sound and informed by the powerful impact the Floors Problem had on their own learning. Also, one can see parallels between PTs' responses in part C and the part B pillars, which were supported by our instructional design.

*Prompt 1:* An illustrative explanation of the first question in prompt 1 was: "You have to start with a base and then make assumptions from that. So your conclusion can only be based on those assumptions. Therefore your conclusion can be only true or false for those assumptions." Regarding the second question in prompt 1, all groups noted the importance of helping students develop a sense of the role of assumptions in mathematics. The groups justified their response with reference to: the importance of assumptions in the discipline; the fact that knowledge of assumptions can help students understand that different legitimate interpretations of a problem can lead to different conclusions; and that knowledge of assumptions can highlight the need for students to explain their thinking. These justifications relate to element 3 and are also underpinned by the understanding of elements 1 and 2 that was evidenced in PTs' responses to the first question in prompt 1.

*Prompt 2:* All groups disagreed with the statement and highlighted, in a similar manner, that the phrasing of a task should align with the teacher's goals. Here is an illustrative response: "If a teacher is giving a math test, or graded evaluation where they are looking for a specific answer then yes, the tasks should be unambiguous. However, if the teacher is trying to prove a point about assumptions or different approaches to the same problem, an ambiguous task may be beneficial."

*Prompt 3:* Two major points emerged from the responses to prompt 3. The first point related to teachers being able to recognize that some students' approaches to a task that appear as "faulty" may be mathematically sound based on unforeseen (to the teacher) legitimate assumptions. The second point related to teachers asking students to explain their thinking to see whether the students indeed operated on different assumptions rather than just focus on the final answer. Again, these justifications relate to element 3 but are also based on understanding of elements 1 and 2.

## Conclusion

In this paper we investigated how teacher education may promote prospective elementary teachers' knowledge about the role of assumptions in mathematical activity, with attention to the design and

implementation of a task sequence that aimed to support three important elements of MKfT in this area. We used the particular task sequence to illustrate an approach to promoting MKfT whereby initial work on a mathematics task can be not only mathematically but also pedagogically generative, with follow-up instruction fostering the intertwinement between mathematics and pedagogy. Analysis of the purposeful implementation of the task sequence we conducted in the last research cycle of our design experiment in a mathematics course for prospective elementary teachers provided evidence for the promise of the task sequence in promoting the three elements of MKfT. A limitation of the study is the absence of a follow up investigation of whether the rich insights the PTs developed about the role of assumptions, both mathematical and pedagogical, informed their classroom teaching. This is a promising direction for future research.

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