University students' understandings of concept relations and preferred representations of continuity and differentiability

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The aim of the study reported in this paper is to investigate how students understand continuity and differentiability during and after a calculus course. The students' choices of representations, both claimed and acted, were also studied. The study is part of a larger study of four student groups taking a calculus course. 207 students answered a questionnaire during the course and of them, 11 were interviewed after the course (the ones in this paper). Answers in questionnaires and interviews were categorised and compared. All students who preferred formal theoretical representations, and only those students, were able to produce formal proofs. The students' stated and acted preferences of representations were quite coherent, with only a few inconsistencies.

Keywords: Calculus, continuity, differentiability, understanding, representations.

Introduction

Learning means adaption of, building on and sometimes rejection of prior knowledge. Calculus at university level comprises numerous new things to learn for many students and the actual learning may take place a while after the teaching occasion or even the examination. Differentiability and continuity, the topics studied in this paper, are closely linked to limits that has been proven difficult to learn (e.g. Juter, 2005, 2012 and Tall & Vinner, 1981). Nagle (2013) concludes, in her overview of research on transitions to formal limit conceptions, that there is a consensus in the results about the students' insufficiently developed concept images that do not allow them to formally understand limits. The transition requires the students to go from a dynamic, discrete way of perceiving limits as processes to a static, continuous viewpoint where limits are regarded as formal objects. Nagle suggests an alternative introduction to calculus where more time is spent on informal conceptions to ease the transition to a formal definition. The exams also influence students' studying strategies. Bergqvist (2007) found that a vast majority of tasks from 16 university exams in introductory calculus from four different universities in Sweden only required imitative reasoning skills to pass. Mathematics learning is then endangered to become reduced to remembering routines rather than understanding concepts, processes and relationships. The study in this paper further investigates formal and informal representations used by students to argue for relational properties of continuity and differentiability during a calculus course, and how they have developed after the course. The students' exams were divided in two, where the first part was a written routine problem solving exam and the second an oral exam where definitions and proofs were assessed. The students' preferred types of representations were also investigated and compared to their understandings to further explain how they learn the concepts. The research questions addressed in the paper are:

How do students' relational understandings of continuity and differentiability during a calculus course compare to their understandings after the course?

How do students' claimed preferences of representations match their actual use of representations?

How do students' understandings and preferences, spoken and acted, of representations correlate?

Theoretical frame and some prior results

Students' understandings of mathematical concepts are reflected in their solutions, reasoning and other actions as traces (Juter, 2005) of their concept images, i.e. the total cognitive representation of a concept that an individual has in his or her mind (Tall, & Vinner, 1981). Tall and Vinner define a person's concept definition for a concept as the words or symbols used to define the concept. Understanding a concept and being able to solve tasks involving the concept may be regarded as synonyms for some students, particularly if being able to solve tasks through imitative reasoning is enough to pass exams. The two ways of dealing with mathematics can however be distinguished according to their core features. Hiebert and Lefevre (1986) defined conceptual knowledge (p. 3) as a web of pieces of information well linked together with meaningful connections. Relations between concepts are abundant and significant. They defined procedural knowledge (p. 6) as knowledge requiring an input which the learner recognises and is able to perform a linear procedure on to obtain an outcome. No relational understanding is required for the process to be carried through. Strong and valid connections between concepts, i.e. conceptual knowledge, help learners to understand more as new information is embedded in, and supported by, their existing knowledge (Hiebert & Carpenter, 1992). Rich connections between concepts also reduce the burden of remembering pieces of knowledge and makes transfer within the concept image easier. Students are often unaware of the quality of links between concepts in their concept images, particularly if irrelevant or untrue links are mixed with true ones (Juter, 2011). A large number of links enables students to explain what they think determines a concept or a relationship between concepts. This can give a false sense of understanding if the links are incorrect, which in turn may lead to a situation where the student is unaware of any need for further work with the concept.

Connections between different representations of the same concept, as well as connections between different concepts, are important to create strong concept images. A function can for example be represented in different ways algebraically, by a graph, or in words. Santi (2011) addressed the issue of students understanding different representations of the same mathematical phenomenon or concept, e.g. tangents. He compared the limit process of a derivative in calculus with a cognitive perspective to a more embodied Euclidean approach of the tangent touching the curve in one point. Some students showed difficulties in seeing those representations as the same object. In a study of university students learning limits of functions (Juter, 2005), another example of incoherence in representations of a concept was apparent. Several students interpreted the formal theory as stating that limits are unattainable for functions, but when limits were used in problems they could see that sometimes functions could attain limit values (e.g. linear functions). When both these perceptions were evoked simultaneously, the students became confused. Students meet different definitions and representations, depending on context, e.g. intuitive descriptions, informal definitions and formal definitions (Jayakody & Zazkis, 2015). Jayakody and Zazkis presented two definitions of continuity based on limit definitions used at university courses. They concluded that students should investigate different definitions and their consequences to better understand the purpose of them. When investigating a function for continuity, the results may differ depending on definition choice, particularly if the definitions are learned intuitively rather than formally. An intuitive representation is here regarded as a perceived self-evident mental representation of a concept or phenomenon, as described by Dreyfus and Eisenberg (1982). An intuitive representation often lacks the benefits of formal strictness that can be useful in particular situations, e.g. determining if a function is continuous in a neighborhood of a given point. Developing conceptual knowledge may be difficult based mainly on intuitive perceptions. In the example with attainability of limits (Juter, 2005) some students misinterpreted the strict inequalities in the formal definition to mean that the function never can attain the limit value. The intuitive interpretation of that part of the definition overthrew the formal definition leaving the students with an incoherent concept image. Intuitive representations work as support for learning in many cases, but sometimes they are obstacles, particularly in a procedural learning approach where there are few opportunities to understand relations from deductive reasoning. In this study students formal, informal and intuitive representations of continuity and differentiability are studied and compared to the students' stated and acted preferences of representation forms.

The study, methods and sample

The 11 students focused on in this study were part of a larger study of 207 students enrolled in their first calculus course at university level. The course was not given in one particular program, so the students were from different disciplines, e.g. physics or mathematics. Their understandings of continuity and differentiability, and proving strategies of statements regarding the concepts, were examined (for prior results see Juter (2012)). The students were from four different groups taking the same course (different semesters). The duration of the course was 10 weeks and included basic calculus with limits, continuity, derivatives, integrals, differential equations and Taylor's formula. The students wrote an individual exam with focus on problem solving, mainly with calculations, at the end of the course and if they passed, they took an individual oral exam covering the theory of the course a couple of days later. The students answered a questionnaire when they had covered continuity and derivatives in the course. The 207 students in the study were all answering the questionnaire, which was more than 90% of the students attending the lectures. They filled it out after a lecture and had as much time as they wanted (they used up to about 30 minutes). The aim was to learn more about the students' understandings of the concepts and the relation between them, but also how they expressed their responses, e.g. formally or intuitively. The questions were for those reasons openly formulated. The first five questions were about what features continuous functions and differentiable functions have and what the concepts are used for. The questions relevant for the part reported here followed and they are:

1. Are all continuous functions differentiable? Justify your answer.
2. Are all differentiable functions continuous? Justify your answer.

The aim with these two questions was to see what types of representations the students would select to argue for their hypotheses. Before the data collection, they had seen examples and proofs that would enable them to answer both questions even though they were differently formulated than in the course. After the course, 11 of the students were individually interviewed. The students volunteered by indications in their questionnaires and were selected from their questionnaire answers to exemplify conceptual understanding, procedural understanding, formal use of theory, informal use of theory and intuitive reasoning. The selected students are described after Table 1. Each interview lasted about 30-45 minutes and was audio recorded. The questions were about the questions from the questionnaire and the students' answers to them, proving, examination forms and attitudes to mathematics. They were particularly asked if they agreed to their former statements in the questionnaire or not. The analysis of the interviews were tightly connected to the questionnaires and the students' development from them. Representation forms as well as mathematical content were analyzed and categorized.

Results and discussion

Table 1 shows the students' answers to the two questions in the questionnaire (Q), if they agree or disagree (correctly or incorrectly) to those answers at the interview (I) after the course, and if the students managed to prove their statement in the second question (if so, in the questionnaire (Q) or the interview (I)).

Stud.	*Continuous implies diff. Q*	*Diff. implies continuous Q*	*Agrees (I) correctly/ incorrectly*	*Disagrees (I) correctly/ incorrectly*	*Proves formally, Q or I*
Jonas	No, $\|x\|$	Yes, small change in x causes a small change in y	Correctly but he wants something added about intervals		Yes, I
Jack	No, $\|x\|$	Yes, no actual reason	Correctly explaining why $\|x\|$ is not differentiable		Yes, I
Jim	No, not $\|x\|$ and endpoints of $[a,b]$	Yes, differentiability is a stronger feature than continuity	Correctly		Yes, I
John	No, $\|x\|$	Yes, correct formal proof using the definitions of continuity and derivative	Correctly		Yes, Q
Felicia	No, $\|x\|$	Yes, same left and right limit, slope independent of chosen point in the neighbourhood of the point	Correctly agrees with first question and explains why $\|x\|$ is not differentiable	Correctly clarifies her answer to the second question. Thinks it was messily formulated	No
Fred	No, $\|x\|$	Yes, no reason	Correctly (some confusion)		No
Fay	Yes, no jumps in a neighbour-hood of an undefined point so same	No, a function may be differentiable on an interval, but not in the actual		Correctly but a bit vaguely justified in a formal attempt	No

	limits form left and right	jump			
Clara	No, only if defined for all points in an interval	No, no reason	Incorrectly		No
Carly	Yes, since they always have a slope	Answers that continuous implies differentiable again	Incorrectly on the first question, not really addressing the second		No
Celia	No, |*x*|	Yes, no actual reason	Agrees but adds error: In (0, 0) is |*x*| not continuous		No
Carl	No, at peaks there are many different tangents. States that continuous implies diff. in another question	No, not a stair function	Correctly agrees on the first question	Correctly disagrees to his statement that continuous implies differentiable but unable to clarify the second question	No

Table 1: Students' understandings of continuity and differentiability from questionnaires during the course (Q) and interviews after the course (I)

The students in Table 1 are categorized in three groups, separated by different first letters in their fictitious names, depending on their responses to the two questions in the questionnaire and the interviews. In the first group (all names start with J), the four students correctly answered the questions in the questionnaires and interviews and came up with correct formal proofs. All four students used $|x|$ as a counter example to show a continuous non-differentiable function in the questionnaires. Three of the four students (all but John) did not prove their answers to the second question in the questionnaires, but they were all able to do so in the interview. Jim did at first not think he was able to prove his statement in the interview, but when he got started he was able to take it deductively step by step through knowledge about the concepts revealing a conceptual (Hiebert & Lefevre, 1986) approach to mathematics in this area. Jack had a similar task to prove at his oral exam and showed confidence in procedurally proving it in the interview, even though he was unable to prove it during the course in the questionnaire. In the second group, with three students, all names start with F. The students either answered correctly at the questionnaire and agreed with their answers in the interview (Fred and Felicia) or answered wrongly at the questionnaire and then disagreed in the interview (Fay). Felicia and Fred both used $|x|$ as a counter example the same way the students in the first group did. The students in the second group could show some confusion or small mistakes, but they answered correctly in a large sense after the course. The students did not produce any proof of the second question, but Fay made an attempt to do so when she was asked to try. She was however unable to see it through after she had written the definition for continuity where x tends to a and the definition for derivative where h tends to 0. It would probably work better for her if she had used a definition of derivative where x tends to a so she could combine the definitions easier, but her concept definitions did not allow such flexibility. Comparisons of various definitions, as suggested by Jayakody and Zazkis (2015), could have helped her adjust her concept definitions to work together. She also thought that a limit is not an exact value, which can lead to problems understanding that a tangent in a point is unique if it exists, as Santi (2011) found. The third group comprises four students, all names starting with a C. These students were unable to correctly answer and/or justify their answers in the interviews. Carl and Carly both stated that continuous functions are differentiable in the questionnaire (Carl wrote it as an answer to another question where he was asked what features continuous functions have, so he gave two opposing answers in the questionnaire since he answered 'no' to question 1 above) and Carly kept that opinion in the interview whereas Carl changed to a correct standpoint. He was however not able to correctly answer the second question. Carly thought that all continuous functions have a specific slope in all points and are hence differentiable. In the interview she thought that all differentiable functions are continuous in an open interval since the tangent does not fall over the edge at the endpoints. Carly had an intuitive (Dreyfus & Eisenberg, 1982), non-formal, way of explaining her thoughts as this example indicates. Celia got the answers correct but added erroneous explanations that did not seem founded in any conceptual knowledge, e.g. $|x|$ is not continuous at (0, 0).

There were various kinds of confusion in all groups, but in the first group it was only Jonas who lacked something about intervals in his own reasoning in the questionnaire, and this was sorted out in the interview. The other two groups showed more serious errors and confusion as described. The clarity in representation varied in the students' responses to questions in the study and the students used different types of representations to argue for their hypotheses. Table 2 show the students' preferred representation styles as they described it and as they acted when answering the questionnaire (Q) and in the interviews (I). The category called "F theory" means students using formally expressed definitions and theorems. "Pictures" is a category for students using diagrams or other figures to explain. "Words" is a category for descriptions in words, including theoretical and intuitive descriptions. The categories can be combined, e.g. "Theory" and "Words" in John's description where he formally stated a proof of the second question using definitions of derivative and continuity and explained the definition of continuity using words. John specifically stated that he did not use pictures ever, Clara stated that she read the formal theoretical parts, but did not use that way to express herself and Celia described her learning intentions to be shallow with no focus on formal theory. The categories for these three students are specified according to this in Table 2.

Students	*Says to prefer in interview*	*Preferences in action*
Jonas	F theory	F theory I, Words Q
Jack	F theory	F theory I, Words Q
Jim	F theory	F theory I, Words Q
John	F theory, Not pictures	F theory I, Q, Words I
Felicia	Pictures	Pictures I, Words Q

Fred	F theory	Pictures I, Q, Words Q
Fay	Pictures	Words I, Q, Pictures Q, F theory I
Clara	Reads F theory	Words I, Q
Carly	Pictures	Pictures I, Words Q
Celia	Not F theory	Words I, Q
Carl	Pictures	Pictures I, Q

Table 2: Students' outspoken and acted preferences of representation forms in interview (I) and questionnaire (Q

There is a rather good correlation between what representations the students said they preferred and what they used in this study. Fred's statement and actions were most apart as he said that he preferred formal theory, but showed no traces of it. Instead he used pictures and words as did Felicia and Fay. These are only narrow timespans to look at the students' mathematics representations so they may of course vary from what is reported here. Female students used pictures in their reasoning to a higher extent than male students did, according mainly to their own statements, but the trend can also be seen in their actions. Carly, who was very visual in her explanations, preferred representations as pictures. Her mathematical development was not conceptually strong as her representations were vague and erroneous. Fay also preferred pictures, but she turned to formal representations when she was urged to try to conduct a proof (as afore described). Male students had a stronger focus on formal theory throughout. All four students in the first group (names starting with J) said to prefer formal theoretical representations and correspondingly used formal theory. John even emphasized that he did not use pictures, which he also did not do in this study. No other student than these four both said to prefer formal representations and used formal representations in justifying claims. The four students were the only ones who could prove that differentiable functions are continuous (Table 1). Three of them were unable to prove it in the questionnaire (or did not do it for other reasons) even though they had just covered the topic in the course, but managed to prove it in the interviews. One reason may be that many students learned the theory after the course for the oral exam since the theory was examined then. If so, they did not use much of the theory in problem solving or in making sense of mathematics during the course.

Celia stands out from the other students in Table 2, as she showed traces of a concept image with quite weak connections. She was aware of the weaknesses since it was her strategy to learn mathematics shallowly and she kept on learning that way on purpose. Her stated approach to mathematics was procedural and she had no attempt to learn anything conceptually. This was also very clear in her responses in the interview and the questionnaire (see Table 1). Celia had a representation of $|x|$ not being differentiable, but she did not know why. She had an intuitive sense of how it should be and kept that standpoint even though she had no means available in her concept image to justify or explain it. When she tried to explain she came to the wrong conclusion that $|x|$ is not continuous at (0, 0).

Conclusions

The changes in students' understandings of continuity and differentiability from the time right after they have learned the concepts (Q) to after the exams (I) were mainly correct adjustments. Some added errors occurred but the main type of changes were improvements of the concept images. It appears as if the students' conceptual understanding and use of theory had matured and small mistakes could be clarified deductively after the course. Students with more serious misunderstandings or insubstantial learning strategies during the course did however not show evidence of understanding the concepts better after the course (e.g. Clara and Carly). Most students' descriptions of what types of representations they used agreed with their actual usages in the data sample. A clear result is that all students in the first group claimed to prefer formal theoretical representations, all used them and all (and only they) were able to correctly prove the statement in the second question. The results of this study imply that further development of conceptual understanding after the learning situation depends on students' preferred representation style. Formal representations seem to be most useful for developing conceptual understanding of the concepts.

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